Generalized r-matrix structure and algebro-geometric solution for integrable systems

Zhijun Qiao[a,b,c]

[a]Institute of Mathematics, Fudan University, Shanghai 200433, P. R. China

[b]Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

[c]Fachbereich 17, Mathematik, Universität-GH Kassel,

Heinrich-Plett-Str. 40, D-34109 Kassel, Germany

Email qiao@lanl.gov zhijun@uni-kassel.de

This paper has been published in Rev. Math. Phys, 13(2001), 545-586

Abstract

The purpose of this paper is to construct a generalized r-matrix structure of finite dimensional systems and an approach to obtain the algebro-geometric solutions of integrable nonlinear evolution equations (NLEEs). Our starting point is a generalized Lax matrix instead of usual Lax pair. The generalized r-matrix structure and Hamiltonian functions are presented on the basis of fundamental Poisson bracket. It can be clearly seen that various nonlinear constrained (c-) and restricted (r-) systems, such as the c-AKNS, c-MKdV, c-Toda, r-Toda, c-Levi, etc, are derived from the reduction of this structure. All these nonlinear systems have r-matrices, and are completely integrable in Liouville's sense. Furthermore, our generalized structure is developed to become an approach to obtain the algebro-geometric solutions of integrable NLEEs. Finally, the two typical examples are considered to illustrate this approach: the infinite or periodic Toda lattice equation and the AKNS equation with the condition of decay at infinity or periodic boundary.

Keywords Lax matrix, r-matrix structure, integrable system, algebro-geometric solution.

MSC 2000: 35Q53, 58F07, 35Q35.

1 Introduction

Completely integrable systems are widespreadly applied in field theory, fluid mechanics, nonlinear optics and other fields of nonlinear sciences. The new development of integrability theory can be roughly divided into three stages. The first one was the direct use of Lax equations for some systems such as the Calogero-Moser system [33] and the Euler rigid equation [32], which were allowed for integration. These systems were not amenable to the classical techniques of integrating Hamiltonian equations of motion. The second one was the so-called 'algebraization', i.e., the tools of Lie algebras, Kac-Moody algebras were used to sysmetically construct a large class of soliton equations and integrable systems [2, 45, 47], and simultaneously present the Lax representations of soliton equations, Hamiltonian structures. The third one is being developed and witnessed through the use of nonlinearization technique [8] to generate finite-dimensional integrable systems. These systems can be the Bargmann system, the C. Neumann system [12, 35], the higher-order constrained flows or symmetric constrained flows [4, 5], and the stationary flows of soliton equations [53]. Indeed, with the help of this method, many new completely integrable systems were successively found [12, 35]. In this way, each integrable system is generated through making nonlinearized procedure for a concrete spectral problem or Lax pair, and it has its own characteristic property. Then a natural question is whether or not there is a unified structure such that it can contain those concrete integrable systems? Recently, the study of r-matrix for nonlinear integrable systems brings a great hope to solving this problem.

Semenov-Tian-Shansky ever gave the definition of r-matrix [47], and used the r-matrix to construct Lie algebra and new Poisson bracket [46] in a given Lie algebra and corresponding coadjoint orbit. The main idea of Semenov-Tian-Shansky and Reyman was how to obtain the new Poisson bracket from a given r-matrix and an element of Lie algebra. Here our thought is how to present r-matrix structure from a given Lax matrix and the standard Poisson bracket:

$$L,\ \{\cdot,\cdot\} \stackrel{?}{\Longrightarrow}\ r - matrix.$$

In the present paper, we give a sure answer for the above question. We propose an approach to generate finite dimensional integrable systems by beginning with the so-called generalized Lax matrix instead of usual Lax pair. Another main result of this paper is to deal with the algebro-geometric solutions of integrable nonlinear evolution equations (NLEEs). It is well-known that the ideal aim for nonlinear equations is to obtain their explicit solutions. According to the nonlinearization method, solutions of integrable NLEEs can have the parametric representations [10] or involutive representations [11], and also have numeric representations in the

discrete case [44]. However, these representations of solutions are not given in an explicit form. Thus, an open question is how to obtain their explicit forms. In the paper we would like to give solutions of integrable NLEEs in the form of algebro-geometric Θ-functions.

The algebro-geometric solutions for some soliton equations with the periodic boundary value problems were known since the works of Lax [28], Dubrovin, Mateev and Novikov [20]. Similar results for the periodic Toda case were obtained slightly later by Date and Tanaka [14]. Afterwards, the relations between commutative rings and ordinary linear periodic differential operators and between algebraic curves and nonlinear periodic difference equations were discussed by Krichever [27]. The technique they adopted is the Bloch eigenfunctions, the spectral theory of linear periodic operators, and some analysis tools on Riemann surfaces.

In the first example of this paper, we give the algebro-geometric solutions of the Toda lattice equation in the infinite or periodic case. Our method is a constraint approach connecting finite dimensional integrable systems with integrable NLEEs instead of usual spectral techniques and Bloch eigenfunctions which are often available to the periodic boundary problems. The results with the periodic boundary conditions are included in ours. In the second example, we consider the well-known AKNS equation. The Ablowitz-Kaup-Newell-Segur (AKNS) equations are a very important hierarchy [1] of NLEEs in soliton theory. It can turn out that the KdV, MKdV, NLS, sine-Gordon, sinh-Gordon equations etc. All these equations are solvable by the inverse scattering transform (IST) [24], and usually have N-soliton solutions [3]. But the algebro-geometric solutions of the AKNS equations have not been obtained since then. We shall solve this problem by using our constraint procedure. The considered AKNS equation is under the case of decay at infinity or periodic boundary condition.

The whole paper is organized as follows. We first introduce a generalized Lax matrix in the next section, then construct a generalized r-matrix structure and a generalized set of involutive Hamiltonian functions in section 3. All those Hamiltonian systems have Lax matrices, r-matrices, and are therefore completely integrable in Liouville's sense. In section 4 it can be seen that various constrained (c-) and restricted (r-) integrable flows, such as the c-AKNS, c-MKdV, c-Toda, r-Toda, c-WKI, c-Levi, etc, can be derived from the reductions of this structure. Moreover, the following interesting facts are given in sections 5, 6, 7, respectively:

– Several pairs of different integrable systems share the same r-matrices with the good property of being non-dynamical (i.e. constant). In particular, a discrete and a continuous dynamical system possess the common Lax matrix, r-matrix, and even completely same involutive set. Additionally, on a symplectic submanifold integrability of the restricted Hamiltonian flow (for continuous case) and symplectic

map (for discrete case) are described by introducing the Dirac-Poisson bracket. They also have the same r-matrix but being dynamical.

– A pair of constrained integrable systems, produced by two gauge equivalent spectral problems, posesses different r-matrices being non-dynamical.

– New integrable systems are generated through choosing new r-matrices from our structure, and the associated spectral problems are also new.

In the last section, as a development of the generalized structure, through considering the relation lifting finite dimensional system to infinite dimensional system and using the algebro-geometric tools we present an approach for obtaining the algebro-geometric solution of integrable NLEEs. To illustrate the procedure we take the periodic or infinite Toda lattice equation and the AKNS equation with the condition of decay at infinity or periodic boundary as the examples.

Before displaying our main results, let us first give some necessary notation: $dp \wedge dq$ stands for the standard symplectic structure in Euclidean space $R^{2N} = \{(p,q)|p = (p_1, \ldots, p_N), q = (q_1, \ldots, q_N)\}$; $< \cdot, \cdot >$ is the standard inner product in R^N; in the symplectic space $(R^{2N}, dp \wedge dq)$ the Poisson bracket of two Hamiltonian functions F, G is defined by [6]

$$\{F, G\} = \sum_{i=1}^{N} (\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}) =< \frac{\partial F}{\partial q}, \frac{\partial G}{\partial p} > - < \frac{\partial F}{\partial p}, \frac{\partial G}{\partial q} >; \tag{1.1}$$

I and $\otimes$ stand for the 2×2 unit matrix and the tensor product of matrix, respectively; $\lambda_1, ..., \lambda_N$ are N arbitrarily given distinct constants; λ, μ are the two different spectral parameters; $\Lambda = diag(\lambda_1, ..., \lambda_N)$, $I_0 =< q, q >$, $J_0 =< p, q >, K_0 =< p, p >$, $I_1 =< \Lambda p, p >< \Lambda q, q >$, $J_1 =< \Lambda p, q >$, a_0, $a_1 = const.$. Denote all infinitely times differentiable functions on real field R by $C^\infty(R)$.

2 A generalized Lax matrix

Consider the following matrix (called Lax matrix)

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix} \tag{2.1}$$

where

$$A(\lambda) = a_{-2}(I_1, J_1)\lambda^{-2} + a_{-1}(J_0)\lambda^{-1} + a_0 + a_1\lambda + \sum_{j=1}^{N} \frac{p_j q_j}{\lambda - \lambda_j}, \tag{2.2}$$

$$B(\lambda) = b_{-1}(I_0, J_0)\lambda^{-1} + b_0(J_0) - \sum_{j=1}^{N} \frac{q_j^2}{\lambda - \lambda_j}, \tag{2.3}$$

$$C(\lambda) = c_{-1}(J_0, K_0)\lambda^{-1} + c_0(J_0) + \sum_{j=1}^{N} \frac{p_j^2}{\lambda - \lambda_j}. \tag{2.4}$$

with some undetermined functions $a_{-2},\ a_{-1},\ b_{-1},\ c_{-1},\ b_0,\ c_0 \in C^\infty(R)$.

Now, in order to produce finite dimensional integrable systems directly from the Lax matrix (2.1), we need an inevitable **Assumption (A)**: $\{A(\lambda), A(\mu)\}$, $\{A(\lambda), B(\mu)\}, \{A(\lambda), C(\mu)\}, \{B(\lambda), B(\mu)\}, \{B(\lambda), C(\mu)\}$, and $\{C(\lambda), C(\mu)\}$ are expressed as some linear combinations of $A(\lambda), A(\mu), B(\lambda), B(\mu), C(\lambda), C(\mu)$ with the cofficients in $C^\infty(R)$. Then we have

Lemma 2.1 *Under* **Assumption (A)**, *$L(\lambda)$ only contains the following cases:*

1. If $a_{-2} \neq const.$, $a_0 = b_0 = c_0 = a_1 = 0,\ a_{-1} = -J_0,\ b_{-1} = I_0$, and $c_{-1} = -K_0$, then a_{-2} satisfies the relation $I_1 = (J_1 + a_{-2})^2 + f(a_{-2})$; if $a_{-2} = const. \neq 0$ and $a_0 = b_0 = c_0 = a_1 = 0$, then $a_{-1} = -J_0,\ b_{-1} = I_0,\ c_{-1} = -K_0$, or $a_{-1} = const.,\ b_{-1} = I_0 + f_1(J_0),\ c_{-1} = -K_0 + g_1(J_0)$, where $f_1,\ g_1$ satisfy the relation $f_1 g_1 = -J_0^2 - 2a_{-1}J_0 + const.$.

2. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = b_0 = c_0 = a_1 = 0$, and $a_0 = const.$

3. $a_{-2} = b_{-1} = a_0 = b_0 = c_0 = a_1 = 0,\ c_{-1} = -K_0$, and a_{-1} satisfies $\frac{da_{-1}}{dJ_0} \neq 0$.

4. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = b_0 = a_1 = 0,\ a_0 = const.$, and $c_0 \neq 0$.

5. $a_{-2} = c_{-1} = b_0 = a_1 = 0,\ a_{-1},\ a_0 = const.,\ b_{-1} = I_0 + g(J_0)$, and c_0 satisfies $\frac{d}{dJ_0}(c_0 g(J_0)) = -2a_0$.

6. $a_{-2} = c_{-1} = a_0 = b_0 = c_0 = a_1 = 0,\ a_{-1} = J_0 + const.$, and $b_{-1} = I_0$.

7. If $a_{-2} = a_0 = b_0 = c_0 = a_1 = 0$, then there are the following five subcases:

(7.1) $a_{-1} = const.,\ c_{-1} = -K_0 + f_2(J_0)$, and $b_{-1} = I_0 + g_2(J_0)$;

(7.2) $a_{-1} = -J_0,\ b_{-1} = I_0$, and $c_{-1} = K_0$;

(7.3) $a_{-1} = -J_0 + const.$, and $b_{-1} = b_{-1}(J_0), c_{-1} = c_{-1}(J_0)$ satisfy $\frac{d}{dJ_0}(b_{-1}c_{-1}) = 2a_{-1}$;

(7.4) $a_{-1} = -J_0 + const.,\ b_{-1} = I_0$, and $c_{-1} = c_{-1}(J_0)$;

(7.5) $a_{-1} = -J_0 + const.,\ c_{-1} = -K_0$, and $b_{-1} = b_{-1}(J_0)$.

8. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = 0,\ a_0,\ a_1 = const.,\ b_0 \neq 0,\ c_0 \neq 0$, and b_0, c_0 satisfy the relation $\frac{d}{dJ_0}(b_0 c_0) = -2a_1$.

9. $a_{-2} = a_{-1} = b_{-1} = c_{-1} = 0,\ c_0,\ a_1,\ a_0 = const.$, and $b_0 \neq 0$.

10. $a_{-2} = b_{-1} = c_0 = a_1 = 0,\ a_{-1},\ a_0 = const.,\ c_{-1} = -K_0 + h(J_0)$, and b_0 satisfies the relation $\frac{d}{dJ_0}(b_0 h(J_0)) = -2a_0$.

The above all functions $f,\ g,\ h,\ f_i,\ g_i\ (i = 1, 2)$ are in $C^\infty(R)$.

Proof Through some calculations we have

$$\begin{aligned}
\{A(\lambda), A(\mu)\} &= 2\frac{\partial a_{-2}}{\partial I_1} < \Lambda p, p > (\frac{\lambda}{\mu^2}B(\lambda) - \frac{\mu}{\lambda^2}B(\mu)) \\
&\quad + 2\frac{\partial a_{-2}}{\partial I_1} < \Lambda q, q > (\frac{\lambda}{\mu^2}C(\lambda) - \frac{\mu}{\lambda^2}C(\mu)) \\
&\quad + 2\frac{\partial a_{-2}}{\partial I_1}(\frac{1}{\lambda^2} - \frac{1}{\mu^2})(< \Lambda p, p > (b_{-1} - I_0) - < \Lambda q, q > (c_{-1} + K_0)) \\
&\quad + 2\frac{\partial a_{-2}}{\partial I_1}(\frac{\mu}{\lambda^2} - \frac{\lambda}{\mu^2})(< \Lambda p, p > b_0 + < \Lambda q, q > c_0),
\end{aligned}$$

$$
\begin{aligned}
\{B(\lambda),B(\mu)\} &= 2\frac{\partial b_{-1}}{\partial J_0}(\frac{1}{\mu}B(\lambda)-\frac{1}{\lambda}B(\mu))+2\frac{db_0}{dJ_0}(B(\lambda)-B(\mu))\\
&+2(\frac{1}{\lambda}-\frac{1}{\mu})(\frac{\partial b_{-1}}{\partial I_0}\frac{db_0}{dJ_0}I_0+b_0\frac{\partial b_{-1}}{\partial J_0}-b_{-1}\frac{db_0}{dJ_0}),\\
\{C(\lambda),C(\mu)\} &= 2\frac{\partial c_{-1}}{\partial J_0}(-\frac{1}{\mu}C(\lambda)+\frac{1}{\lambda}C(\mu))+2\frac{dc_0}{dJ_0}(-C(\lambda)+C(\mu))\\
&+2(-\frac{1}{\lambda}+\frac{1}{\mu})(\frac{\partial c_{-1}}{\partial K_0}\frac{dc_0}{dJ_0}K_0+c_0\frac{\partial c_{-1}}{\partial J_0}-c_{-1}\frac{dc_0}{dJ_0}),\\
\{A(\lambda),B(\mu)\} &= \frac{2}{\lambda-\mu}(-B(\mu)+B(\lambda))-\frac{2}{\lambda}\frac{da_{-1}}{dJ_0}B(\mu)\\
&+\frac{2}{\mu}\frac{\partial b_{-1}}{\partial I_0}B(\lambda)-\frac{2\mu}{\lambda^2}(\frac{\partial a_{-2}}{\partial J_1}B(\mu)+2\frac{\partial a_{-2}}{\partial I_1}<\Lambda q,q>A(\mu))\\
&-2<\Lambda q,q>(2\frac{\partial b_{-1}}{\partial I_0}\frac{\partial a_{-2}}{\partial I_1}J_1+\frac{\partial b_{-1}}{\partial I_0}\frac{\partial a_{-2}}{\partial J_1}+2a_{-2}\frac{\partial a_{-2}}{\partial I_1})\lambda^{-2}\mu^{-1}\\
&+2(-\frac{\partial b_{-1}}{\partial I_0}\frac{da_{-1}}{dJ_0}I_0-b_{-1}\frac{\partial b_{-1}}{\partial I_0}+b_{-1}\frac{da_{-1}}{dJ_0}+b_{-1})\lambda^{-1}\mu^{-1}\\
&-2(2<\Lambda q,q>\frac{\partial a_{-2}}{\partial I_1}(J_0+a_{-1})+\frac{\partial a_{-2}}{\partial J_1}(I_0-b_{-1}))\lambda^{-2}\\
&-2(-b_0\frac{\partial a_{-2}}{\partial J_1}+2a_0<\Lambda q,q>\frac{\partial a_{-2}}{\partial I_1})\lambda^{-2}\mu\\
&+2b_0\frac{da_{-1}}{dJ_0}\lambda^{-1}-2b_0\frac{\partial b_{-1}}{\partial I_0}\mu^{-1}-4a_1<\Lambda q,q>\frac{\partial a_{-2}}{\partial I_1}\lambda^{-2}\mu^2,\\
\{A(\lambda),C(\mu)\} &= \frac{2}{\lambda-\mu}(C(\mu)-C(\lambda))+\frac{2}{\lambda}\frac{da_{-1}}{dJ_0}C(\mu)\\
&+\frac{2}{\mu}\frac{\partial c_{-1}}{\partial K_0}C(\lambda)+\frac{2\mu}{\lambda^2}(\frac{\partial a_{-2}}{\partial J_1}C(\mu)+2\frac{\partial a_{-2}}{\partial I_1}<\Lambda p,p>A(\mu))\\
&+2<\Lambda p,p>(2\frac{\partial c_{-1}}{\partial K_0}\frac{\partial a_{-2}}{\partial I_1}J_1+\frac{\partial c_{-1}}{\partial K_0}\frac{\partial a_{-2}}{\partial J_1}-2a_{-2}\frac{\partial a_{-2}}{\partial I_1})\lambda^{-2}\mu^{-1}\\
&+2(\frac{\partial c_{-1}}{\partial K_0}\frac{da_{-1}}{dJ_0}K_0-c_{-1}\frac{\partial c_{-1}}{\partial K_0}-c_{-1}\frac{da_{-1}}{dJ_0}-c_{-1})\lambda^{-1}\mu^{-1}\\
&-2(-2<\Lambda p,p>\frac{\partial a_{-2}}{\partial I_1}(J_0+a_{-1})+\frac{\partial a_{-2}}{\partial J_1}(K_0+c_{-1}))\lambda^{-2}\\
&-2(c_0\frac{\partial a_{-2}}{\partial J_1}+2a_0<\Lambda p,p>\frac{\partial a_{-2}}{\partial I_1})\lambda^{-2}\mu\\
&-2c_0\frac{da_{-1}}{dJ_0}\lambda^{-1}-2c_0\frac{\partial c_{-1}}{\partial K_0}\mu^{-1}-4a_1<\Lambda p,p>\frac{\partial a_{-2}}{\partial I_1}\lambda^{-2}\mu^2,\\
\{B(\lambda),C(\mu)\} &= \frac{4}{\lambda-\mu}(-A(\mu)+A(\lambda))+\frac{4}{\lambda}\frac{\partial b_{-1}}{\partial I_0}A(\mu)\\
&-\frac{4}{\mu}\frac{\partial c_{-1}}{\partial K_0}A(\lambda)+2(\frac{1}{\mu}\frac{\partial c_{-1}}{\partial J_0}+\frac{dc_0}{dJ_0})B(\lambda)+2(\frac{1}{\lambda}\frac{\partial b_{-1}}{\partial J_0}+\frac{db_0}{dJ_0})C(\mu)\\
&+4(-\frac{\partial b_{-1}}{\partial I_0}+1)a_{-2}\mu^{-2}\lambda^{-1}+4(\frac{\partial c_{-1}}{\partial K_0}+1)a_{-2}\lambda^{-2}\mu^{-1}\\
&+2(\frac{\partial c_{-1}}{\partial J_0}\frac{\partial b_{-1}}{\partial I_0}I_0+2\frac{\partial b_{-1}}{\partial I_0}\frac{\partial c_{-1}}{\partial K_0}J_0+\frac{\partial b_{-1}}{\partial J_0}\frac{\partial c_{-1}}{\partial K_0}K_0
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial b_{-1}}{\partial J_0}c_{-1}-\frac{\partial c_{-1}}{\partial J_0}b_{-1}-2\frac{\partial b_{-1}}{\partial I_0}a_{-1}+2\frac{\partial c_{-1}}{\partial K_0}a_{-1}+2a_{-1})\lambda^{-1}\mu^{-1}\\
&+2(-\frac{\partial b_{-1}}{\partial J_0}c_0+\frac{\partial b_{-1}}{\partial I_0}\frac{dc_0}{dJ_0}I_0-\frac{dc_0}{dJ_0}b_{-1}-2\frac{\partial b_{-1}}{\partial I_0}a_0)\lambda^{-1}\\
&+2(-\frac{\partial c_{-1}}{\partial J_0}b_0+\frac{\partial c_{-1}}{\partial K_0}\frac{db_0}{dJ_0}K_0-\frac{db_0}{dJ_0}c_{-1}+2\frac{\partial c_{-1}}{\partial K_0}a_0)\mu^{-1}\\
&-2(\frac{db_0}{dJ_0}c_0+\frac{dc_0}{dJ_0}b_0+2a_1)+4\frac{\partial c_{-1}}{\partial K_0}a_1\mu^{-1}\lambda-4\frac{\partial b_{-1}}{\partial I_0}a_1\lambda^{-1}\mu.
\end{aligned}
$$

According to **Assumption (A)**, the terms that do not contain $A(\lambda),\ A(\mu),\ B(\lambda),\ B(\mu), C(\lambda),\ C(\mu)$ in the above six equalities, are zero. After discussing these terms, we can obtain every result in Lemma 2.1. ∎

3 Generalized r-matrix structure and integrable Hamiltonian systems

Let $L_1(\lambda)=L(\lambda)\otimes I,\ L_2(\mu)=I\otimes L(\mu)$. In the following, we search for a general 4×4 r-matrix structure $r_{12}(\lambda,\mu)$ such that the fundamental Poisson bracket [21]:

$$
\{L(\lambda)\overset{\otimes}{,}L(\mu)\}=[r_{12}(\lambda,\mu),L_1(\lambda)]-[r_{21}(\mu,\lambda),L_2(\mu)] \tag{3.1}
$$

holds, where $r_{21}(\lambda,\mu)=Pr_{12}(\lambda,\mu)P,\ P=\frac{1}{2}\sum_{i=0}^{3}\sigma_i\otimes\sigma_i$, and $\sigma_i's$ are the standard Pauli matrices. For the given Lax matrix (2.1) and the Poisson bracket (1.1), we have the following Theorem.

Theorem 3.1 *Under* **Assumption (A)**,

$$
r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+S \tag{3.2}
$$

is an r-matrix structure satisfying (3.1), where

$$
S=\begin{pmatrix}
\frac{2\lambda}{\mu^2}\frac{\partial a_{-2}}{\partial J_1}+\frac{2}{\mu}\frac{da_{-1}}{dJ_0} & \frac{2}{\mu}\frac{\partial b_{-1}}{\partial J_0} & \frac{2\lambda}{\mu^2}<\Lambda q,q>\frac{\partial a_{-2}}{\partial I_1} & 0\\
2\frac{dc_0}{dJ_0} & 0 & \frac{2}{\mu}\frac{\partial c_{-1}}{\partial K_0} & -\frac{2\lambda}{\mu^2}<\Lambda q,q>\frac{\partial a_{-2}}{\partial I_1}\\
-\frac{2\lambda}{\mu^2}<\Lambda p,p>\frac{\partial a_{-2}}{\partial I_1} & -\frac{2}{\mu}\frac{\partial b_{-1}}{\partial I_0} & 0 & -2\frac{db_0}{dJ_0}\\
0 & \frac{2\lambda}{\mu^2}<\Lambda p,p>\frac{\partial a_{-2}}{\partial I_1} & -\frac{2}{\mu}\frac{\partial c_{-1}}{\partial J_0} & \frac{2\lambda}{\mu^2}\frac{\partial a_{-2}}{\partial J_1}+\frac{2}{\mu}\frac{da_{-1}}{dJ_0}
\end{pmatrix}.
$$

Proof Under **Assumption (A)**, we have

$$
\begin{aligned}
\{A(\lambda),A(\mu)\} &= 2\frac{\partial a_{-2}}{\partial I_1}<\Lambda p,p>(\frac{\lambda}{\mu^2}B(\lambda)-\frac{\mu}{\lambda^2}B(\mu))\\
&\quad+2\frac{\partial a_{-2}}{\partial I_1}<\Lambda q,q>(\frac{\lambda}{\mu^2}C(\lambda)-\frac{\mu}{\lambda^2}C(\mu)),
\end{aligned}
$$

$$
\begin{aligned}
\{B(\lambda), B(\mu)\} &= 2\frac{\partial b_{-1}}{\partial J_0}(\frac{1}{\mu}B(\lambda) - \frac{1}{\lambda}B(\mu)) + 2\frac{db_0}{dJ_0}(B(\lambda) - B(\mu)), \\
\{C(\lambda), C(\mu)\} &= 2\frac{\partial c_{-1}}{\partial J_0}(-\frac{1}{\mu}C(\lambda) + \frac{1}{\lambda}C(\mu)) + 2\frac{dc_0}{dJ_0}(-C(\lambda) + C(\mu)), \\
\{A(\lambda), B(\mu)\} &= \frac{2}{\lambda - \mu}(-B(\mu) + B(\lambda)) - \frac{2}{\lambda}\frac{da_{-1}}{dJ_0}B(\mu) + \frac{2}{\mu}\frac{\partial b_{-1}}{\partial I_0}B(\lambda) \\
&\quad -\frac{2\mu}{\lambda^2}(\frac{\partial a_{-2}}{\partial J_1}B(\mu) - 2\frac{\partial a_{-2}}{\partial I_1} < \Lambda q, q > A(\mu)), \\
\{A(\lambda), C(\mu)\} &= \frac{2}{\lambda - \mu}(C(\mu) - C(\lambda)) + \frac{2}{\lambda}\frac{da_{-1}}{dJ_0}C(\mu) + \frac{2}{\mu}\frac{\partial c_{-1}}{\partial K_0}C(\lambda) \\
&\quad +\frac{2\mu}{\lambda^2}(\frac{\partial a_{-2}}{\partial J_1}C(\mu) + 2\frac{\partial a_{-2}}{\partial I_1} < \Lambda p, p > A(\mu)), \\
\{B(\lambda), C(\mu)\} &= \frac{4}{\lambda - \mu}(-A(\mu) + A(\lambda)) + \frac{4}{\lambda}\frac{\partial b_{-1}}{\partial I_0}A(\mu) - \frac{4}{\mu}\frac{\partial c_{-1}}{\partial K_0}A(\lambda) \\
&\quad +2(\frac{1}{\mu}\frac{\partial c_{-1}}{\partial J_0} + \frac{dc_0}{dJ_0})B(\lambda) + 2(\frac{1}{\lambda}\frac{\partial b_{-1}}{\partial J_0} + \frac{db_0}{dJ_0})C(\mu).
\end{aligned}
$$

which completes the proof. ∎

In general, Eq. (3.2) is a dynamical r-matrix structure, i.e. dependent on canonical variables p_i, q_i [7].

Now, we turn to consider the determinant of $L(\lambda)$

$$
\begin{aligned}
-\det L(\lambda) &= \frac{1}{2}TrL^2(\lambda) = A^2(\lambda) + B(\lambda)C(\lambda) \\
&= \sum_{i=-4}^{2} H_i\lambda^i + \sum_{j=1}^{N}\frac{E_j}{\lambda - \lambda_j}
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
H_{-4} &= a_{-2}^2, &(3.4)\\
H_{-3} &= 2a_{-2}a_{-1}, &(3.5)\\
H_{-2} &= a_{-1}^2 + 2a_{-2}a_0 + b_{-1}c_{-1} - 2a_{-2} < \Lambda^{-1}p, q >, &(3.6)\\
H_{-1} &= 2a_{-2}a_1 + 2a_{-1}a_0 + b_{-1}c_0 + b_0c_{-1} - 2a_{-2} < \Lambda^{-2}p, q > \\
&\quad -2a_{-1} < \Lambda^{-1}p, q > -b_{-1} < \Lambda^{-1}p, p > +c_{-1} < \Lambda^{-1}q, q >, &(3.7)\\
H_0 &= a_0^2 + 2a_{-1}a_1 + b_0c_0 + 2a_1 < p, q >, &(3.8)\\
H_1 &= 2a_0a_1, &(3.9)\\
H_2 &= a_1, &(3.10)\\
E_j &= (2a_{-2}\lambda_j^{-2} + 2a_{-1}\lambda_j^{-1} + 2a_0 + 2a_1\lambda_j)p_jq_j \\
&\quad +(b_{-1}\lambda_j^{-1} + b_0)p_j^2 - (c_{-1}\lambda_j^{-1} + c_0)q_j^2 - \Gamma_j, &(3.11)\\
&\quad \Gamma_j = \sum_{k=1,k\neq j}^{N}\frac{(p_jq_k - p_kq_j)^2}{\lambda_j - \lambda_k},\ j = 1, 2, ..., N. &(3.12)
\end{aligned}
$$

Let (3.3) be multiplied by a fixed multiplier λ^k ($k \in Z$), then it leads to

$$\begin{aligned}
\frac{1}{2}\lambda^k \cdot TrL^2(\lambda) &= \sum_{l=-4}^{2} H_l\lambda^{l+k} + \sum_{i=0}^{k-1} F_i\lambda^{k-1-i} + \sum_{j=1}^{N} \frac{\lambda_j^k E_j}{\lambda-\lambda_j} \\
&= \sum_{l=k-4}^{-1} H_{l-k}\lambda^l + \sum_{l=0}^{k-1}(H_{l-k}+F_{k-1-l})\lambda^l \\
&\quad + \sum_{l=k}^{k+2} H_{l-k}\lambda^l + \sum_{j=1}^{N} \frac{\lambda_j^k E_j}{\lambda-\lambda_j}
\end{aligned} \tag{3.13}$$

where

$$F_m = \sum_{j=1}^{N} \lambda_j^m E_j, \;\; m = 0,1,2,\ldots \tag{3.14}$$

which reads

$$\begin{aligned}
F_m &= 2a_{-2} < \Lambda^{m-2}p, q > +2a_{-1} < \Lambda^{m-1}p, q > +2a_0 < \Lambda^{m}p, q > +2a_1 < \Lambda^{m+1}p, q > \\
&\quad b_{-1} < \lambda^{m-1}p, p > +b_0 < \Lambda^{m}p, p > -c_{-1} < \Lambda^{m-1}q, q > -c_0 < \Lambda^{m}q, q > \\
&\quad - \sum_{i+j=m-1} (< \Lambda^i p, p >< \Lambda^j q, q > - < \Lambda^i p, q >< \Lambda^j p, q >).
\end{aligned} \tag{3.15}$$

Because there is an r-matrix structure satisfying (3.1),

$$\{L^2(\lambda) \overset{\otimes}{,} L^2(\mu)\} = [\bar{r}_{12}(\lambda,\mu), L_1(\lambda)] - [\bar{r}_{21}(\mu,\lambda), L_2(\mu)], \tag{3.16}$$

where

$$\bar{r}_{ij}(\lambda,\mu) = \sum_{k=0}^{1}\sum_{l=0}^{1} L_1^{1-k}(\lambda)L_2^{1-l}(\mu)\cdot r_{ij}(\lambda,\mu)\cdot L_1^k(\lambda)L_2^l(\mu), \quad ij = 12, 21. \tag{3.17}$$

Thus,

$$4\{TrL^2(\lambda), TrL^2(\mu)\} = Tr\{L^2(\lambda) \overset{\otimes}{,} L^2(\mu)\} = 0. \tag{3.18}$$

So, by (3.13) we immediately obtain

Theorem 3.2 *Under* **assumption (A)**, *the following equalities*

$$\begin{aligned}
&\{E_i, E_j\} = 0, \;\; \{H_l, E_j\} = 0, \;\; \{F_m, E_j\} = 0, \\
&i, j = 1, 2, \ldots, N, \;\; l = -4, \ldots, 2, \;\; m = 0, 1, 2, \ldots,
\end{aligned} \tag{3.19}$$

hold. Hence, the Hamiltonian systems (H_l) *and* (F_m)

$$(H_l) \quad : \qquad q_x = \frac{\partial H_l}{\partial p}, \;\; p_x = -\frac{\partial H_l}{\partial q}, \;\; l = -4, \ldots, 2, \tag{3.20}$$

$$(F_m) \quad : \qquad q_{t_m} = \frac{\partial F_m}{\partial p}, \;\; p_{t_m} = -\frac{\partial F_m}{\partial q}, \quad m = 0, 1, 2, \ldots, \tag{3.21}$$

are completely integrable in Liouville's sense.

Corollary 3.1 *All composition functions* $f(H_l, F_m)$, $f \in C^\infty(R)$ *are completely integrable Hamiltonians in Liouville's sense.*

4 Reductions

For the various cases of Lemma 2.1, we give the corresponding reductions of r-matrix structure $r_{12}(\lambda,\mu)$ in this section. The following numbers of title coincide with the ones in Lemma 2.1, i.e. the corresponding conditions are coincidental.

Before giving our reductions, we'd like to re-stress the two "terminologies" used usually in the theory of integrable systems in order to avoid some confusions: one is "*constrained system*", which means the finite dimensional Hamiltonian system or symplectic map in R^{2N} under the *Bargmann-type* constraint; the other "*restricted system*", which means the finite dimensional Hamiltonian system or symplectic map on some symplectic submanifold in R^{2N} under the *Neumann-type* constraint. In the future we shall follow this principle.

1.

$$r_{12}(\lambda,\mu)=\frac{2\lambda}{\mu(\mu-\lambda)}P+2\frac{\partial a_{-2}}{\partial J_1}\cdot\frac{\lambda}{\mu^2}S+2\frac{\partial a_{-2}}{\partial I_1}\cdot\frac{\lambda}{\mu^2}Q, \tag{4.1}$$

$$S=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\quad Q=\begin{pmatrix}0&0&<\Lambda q,q>&0\\0&0&0&-<\Lambda q,q>\\-<\Lambda p,p>&0&0&0\\0&<\Lambda p,p>&0&0\end{pmatrix}.$$

Particularly, with $f(a_{-2})=-1$, (4.1) exactly reads as the r-matrix of the constrained WKI (c-WKI) system [38]. With $a_{-2}=const.\neq 0$, (3.2) reads the r-matrix $r_{12}(\lambda,\mu)$ $=\frac{2\lambda}{\mu(\mu-\lambda)}P$ of ellipsoid geodesic flow [26], or reads as

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+\frac{2}{\mu}S,\quad S=\begin{pmatrix}0&f_1'&0&0\\0&0&-1&0\\0&-1&0&0\\0&0&-g_1'&0\end{pmatrix},$$

which is a new r-matrix structure. For simplicity, below write '$\prime$'$=\frac{d}{dJ_0}$.

2.

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P. \tag{4.2}$$

This is nothing but the r-matrix of the well-known constrained AKNS (c-AKNS) system [8].

3.

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+\frac{2}{\mu}S,\quad S=\begin{pmatrix}a_{-1}'&0&0&0\\0&0&-1&0\\0&0&0&0\\0&0&0&a_{-1}'\end{pmatrix},\ a_{-1}'\neq 0. \tag{4.3}$$

In particular, with $a_{-1} = -J_0$ Eq. (4.3) reads as the r-matrix of the constrained LZ (c-LZ) system [12].

4.

$$r_{12} = \frac{2}{\mu - \lambda} P + c_0' S, \quad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.4}$$

With $c_0 = -2\sqrt{J_0}$, Eq. (4.4) reads as the r-matrix of the constrained Hu (c-H) system [12].

5.

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + S, \quad S = \begin{pmatrix} 0 & \frac{1}{\mu} g' & 0 & 0 \\ c_0' & 0 & 0 & 0 \\ 0 & -\frac{2}{\mu} & 0 & -\frac{1}{\mu} g' \\ 0 & 0 & c_0' & 0 \end{pmatrix}. \tag{4.5}$$

With $b_{-1} = I_0,\ c_0 = const.$, Eq. (4.5) reads as the r-matrix of the constrained Qiao (c-Q) system [39].

6.

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + \frac{2}{\mu} S, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.6}$$

This is a new r-matrix.

7.

(7.1)

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + \frac{2}{\mu} S, \quad S = \begin{pmatrix} 0 & g_2' & 0 & 0 \\ f_2' & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

This is also a new r-matrix.

(7.2)

$$r_{12}(\lambda, \mu) = \frac{2\lambda}{\mu(\mu - \lambda)} P. \tag{4.8}$$

This is the r-matrix of the constrained Heisenberg spin chain (c-HSC) system [40].

(7.3)

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P - \frac{2}{\mu} S, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{-1}' \\ 0 & 0 & c_{-1}' & 1 \end{pmatrix}. \tag{4.9}$$

With $b'_{-1} = -1,\ c'_{-1} = 1$, Eq. (4.9) becomes the r-matrix of the constrained Levi (c-L) system [42].

(7.4)

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P - \frac{2}{\mu}S, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c'_{-1} & 1 \end{pmatrix}. \tag{4.10}$$

This is a new r-matrix.

(7.5)

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P - \frac{2}{\mu}S, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b'_{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

This is also a new r-matrix.

8.

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P + S, \quad S = \begin{pmatrix} 0 & 2b'_0 & 0 & 0 \\ 2c'_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

With $b_0 = c_0 = \sqrt{J_0},\ a_1 = -\frac{1}{2}$, Eq. (4.12) reads as the r-matrix of the constrained Tu (c-T) system [12].

9.

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P + S, \quad S = b'_0 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.13}$$

With $b_0 = -J_0$, Eq. (4.13) reads as the common r-matrix of the constrained Toda (c-Toda) system (a discrete system) and the constrained CKdV (c-CKdV) system (a continuous system), which can be seen in subsection 5.1.

10.

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P + S, \quad S = \begin{pmatrix} 0 & b'_0 & 0 & 0 \\ \frac{1}{\mu}h' & 0 & -\frac{2}{\mu} & 0 \\ 0 & 0 & 0 & -b'_0 \\ 0 & 0 & -\frac{1}{\mu}h' & 0 \end{pmatrix}. \tag{4.14}$$

With $h = const.,\ b_0 = 0$, Eq. (4.14) reads as the r-matrix of the constrained MKdV (c-MKdV) system [37].

Proof For simplicity, we only present the proof in cases 1 and 10, other cases are similar.

Case 1: With $a_{-2} \neq const.$, the matrix S becomes

$$S = \begin{pmatrix} \frac{2\lambda}{\mu^2}\frac{\partial a_{-2}}{\partial J_1} - \frac{2}{\mu} & 0 & \frac{2\lambda}{\mu^2} < \Lambda q, q > \frac{\partial a_{-2}}{\partial I_1} & 0 \\ 0 & 0 & -\frac{2}{\mu} & -\frac{2\lambda}{\mu^2} < \Lambda q, q > \frac{\partial a_{-2}}{\partial I_1} \\ -\frac{2\lambda}{\mu^2} < \Lambda p, p > \frac{\partial a_{-2}}{\partial I_1} & -\frac{2}{\mu} & 0 & 0 \\ 0 & \frac{2\lambda}{\mu^2} < \Lambda p, p > \frac{\partial a_{-2}}{\partial I_1} & 0 & \frac{2\lambda}{\mu^2}\frac{\partial a_{-2}}{\partial J_1} - \frac{2}{\mu} \end{pmatrix}.$$

Substituting S into (3.2) and sorting it, we can obtain (4.1), where a_{-2} satisfies the relation $I_1 = (J_1 + a_{-2})^2 + f(a_{-2})$, for any $f(a_{-2}) \in C^\infty(R)$. Particularly, choosing $f(a_{-2}) = -1$ yields $a_{-2} = \sqrt{1+ < \Lambda p, p >< \Lambda q, q >} - < \Lambda p, q >$. Thus Eq. (4.1) reads

$$r_{12}(\lambda, \mu) = \frac{2\lambda}{\mu(\mu - \lambda)} P - \frac{2\lambda}{\mu^2} S + \frac{\lambda}{\mu^2} \frac{1}{\sqrt{1+ < \Lambda p, p >< \Lambda q, q >}} Q, \qquad (r - WKI)$$

while the corresponding Lax matrix $L(\lambda)$ becomes

$$L(\lambda) = \begin{pmatrix} l_{11} & < q, q > \lambda^{-1} \\ - < p, p > \lambda^{-1} & -l_{11} \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} p_j q_j & -q_j^2 \\ p_j^2 & -p_j q_j \end{pmatrix}$$

where

$$l_{11} = (\sqrt{1+ < \Lambda p, p >< \Lambda q, q >} - < \Lambda p, q >)\lambda^{-2} - < p, q > \lambda^{-1}.$$

Set an auxiliary matrix M_1 as follows

$$M_1 = M_1(\lambda) = \begin{pmatrix} -\lambda & \frac{<\Lambda q, q>}{\sqrt{1+<\Lambda p, p><\Lambda q, q>}}\lambda \\ -\frac{<\Lambda p, p>}{\sqrt{1+<\Lambda p, p><\Lambda q, q>}}\lambda & \lambda \end{pmatrix},$$

then the Lax equation

$$L_x = [M_1, L]$$

is equivalent to the following finite dimensional Hamilton system $(\sqrt{H_{-4}})$:

$$(\sqrt{H_{-4}}): \qquad \begin{cases} q_x = -\Lambda q + \frac{<\Lambda q, q>}{\sqrt{1+<\Lambda p, p><\Lambda q, q>}}\Lambda p = \frac{\partial \sqrt{H_{-4}}}{\partial p}, \\ p_x = \Lambda p - \frac{<\Lambda p, p>}{\sqrt{1+<\Lambda p, p><\Lambda q, q>}}\Lambda q = -\frac{\partial \sqrt{H_{-4}}}{\partial q}, \end{cases} \qquad (4.15)$$

with

$$\sqrt{H_{-4}} = a_{-2} = - < \Lambda p, q > + \sqrt{1+ < \Lambda p, p >< \Lambda q, q >},$$

which is obviously integrable by Theorem 3.2.

Let

$$u = \frac{< \Lambda q, q >}{\sqrt{1+ < \Lambda p, p >< \Lambda q, q >}}, \; v = -\frac{< \Lambda p, p >}{\sqrt{1+ < \Lambda p, p >< \Lambda q, q >}}, \qquad (4.16)$$

then $(\sqrt{H_{-4}})$ is nothing but the WKI spectral problem [51]

$$y_x = \begin{pmatrix} -\lambda & \lambda u \\ \lambda v & \lambda \end{pmatrix} y$$

with the above two constraints (4.16), $\lambda = \lambda_j$, and $y = (q_j, p_j)^T,\ j = 1, \ldots, N$. That means that $(r - WKI)$ is the r-matrix of the integrable constrained WKI (c-WKI) system (4.15).

Other subcases in case 1 can be similarly proven.

The proof of Case 10 can be found in ref. [37]. In this case, the corresponding constrained system is reduced to the well-known MKdV spectral problem [50]. ∎

Remark 4.1 From the above formulas (4.1)-(4.14), the r-matrices of cases 2, 6, and (7.2) are non-dynamical. But in fact, for other cases we can also obtain non-dynamical r-matrices if choosing some special functions, for instance, in Eq. (4.3) setting a_{-1} such that $a'_{-1} = const.$ leads to a non-dynamical one. Of course, we can also get dynamical r-matrix, for instance, in Eq. (4.4) choosing $c_0 = -2\sqrt{J_0}$ yields a dynamical one.

Remark 4.2 Equalities (4.1)-(4.14) cover most r-matrices of 2×2 constrained systems. But among them there are also some new r-matrices and finite dimensional integrable systems like cases 6, 7.1, 7.4, and 7.5 (also see section 7). Their Lax matrices are altogether unified in Eq. (2.1). So, quite a large number of finite dimensional integrable systems are classified or reduced from the viewpoint of Lax matrix and r-matrix structure.

5 Different systems sharing the same r-matrices

In the above r-matrices, we find some pairs of different integrable systems sharing the common r-matrices. Now, we present these results as follows.

5.1 The constrained Toda and CKdV flows

Let us consider the following 2×2 traceless Lax matrix [36] (corresponding to case 9 in section 4)

$$\begin{aligned} L^{TC} &= L^{TC}(\lambda, p, q) = \begin{pmatrix} -\frac{1}{2}\lambda & <p, q> \\ -1 & \frac{1}{2}\lambda \end{pmatrix} + L_0 \\ &\equiv \begin{pmatrix} A_{TC}(\lambda) & B_{TC}(\lambda) \\ C_{TC}(\lambda) & -A_{TC}(\lambda) \end{pmatrix} \end{aligned} \tag{5.1}$$

where

$$L_0 = L_0(\lambda, p, q) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} -p_j q_j & p_j^2 \\ -q_j^2 & p_j q_j \end{pmatrix}. \tag{5.2}$$

The determinant of Eq. (5.1) leads to

$$\frac{1}{2}\lambda Tr(L^{TC})^2(\lambda) = -\frac{1}{2}\lambda^3 + <p,q>\lambda + 2H_C + \sum_{j=1}^{N}\frac{E_j^{TC}}{\lambda-\lambda_j}, \tag{5.3}$$

$$E_j^{TC} = \lambda_j p_j q_j - p_j^2 - <p,q> q_j^2 - \Gamma_j,\ j=1,2,...,N, \tag{5.4}$$

where Γ_j is defined by (3.12) and the Hamiltonian function H_C is

$$H_C = -\frac{1}{2}<p,p> + \frac{1}{2}<\Lambda q,p> - \frac{1}{2}<q,q><p,q>. \tag{5.5}$$

Viewing the variables q and p as the functions of continuous variables x, then we have the following Hamiltonian canonical equation (H_C):

$$\left\{\begin{array}{l} p_x = -\frac{\partial H_C}{\partial q} = -\frac{1}{2}\Lambda p + \frac{1}{2}<q,q>p + <p,q>q, \\ q_x = \frac{\partial H_C}{\partial p} = -p + \frac{1}{2}\Lambda q - \frac{1}{2}<q,q>q, \end{array}\right. \tag{5.6}$$

which is nothing but the coupled KdV (CKdV) spectral problem [29]

$$\psi_x = \begin{pmatrix} -\frac{1}{2}\lambda + \frac{1}{2}u & v \\ -1 & \frac{1}{2}\lambda - \frac{1}{2}u \end{pmatrix}\psi \tag{5.7}$$

with the two constraints (Bargmann-type)

$$u = <q,q>,\quad v = <p,q>, \tag{5.8}$$

$\lambda = \lambda_j$ and $\psi = (p_j, q_j)^T$. So, (H_C) coincides with the constrained CKdV (c-CKdV) flow.

Let us consider endowing with an auxiliary 2×2 matrix M_T as follows

$$M_T = \begin{pmatrix} 0 & g \\ -\frac{1}{g} & \frac{\lambda - <q,q>}{g} \end{pmatrix},\quad g^2 = <\Lambda q,q> - <p,q> - <q,q>^2. \tag{5.9}$$

Then, we have the following theorem.

Theorem 5.1.2 *The discrete Lax equation*

$$(L^{TC})'M_T = M_T L^{TC},\quad (L^{TC})' = L^{TC}(\lambda, p', q') \tag{5.10}$$

is equivalent to a finite-dimensional symplectic map $H_T:\ R^{2N} \longrightarrow R^{2N}, (p,q) \longmapsto (p',q')$, *which is called the constrained Toda (c-Toda) flow*:

$$\left\{\begin{array}{l} p' = gq, \\ q' = \frac{\Lambda q - p - <q,q>q}{g}. \end{array}\right. \tag{5.11}$$

Proof Directly calculate, and readily show (5.10) $\Longleftrightarrow$ (5.11) and $(H_T)^*(dp\wedge dq) = dp\wedge dq$. ∎

When we understand the above two matrices $(L^{TC})'$ and M_T in the following sense: $(L^{TC})' \longrightarrow L^{TC}_{n+1}$, $M_T \longrightarrow M_{Tn}$ (i.e., $q \longrightarrow q_n, p \longrightarrow p_n$, here n is the discrete variable), and set

$$\begin{cases} u_n = \pm(<\Lambda q_n, q_n> - <p_n, q_n> - <q_n, q_n>^2)^{\frac{1}{2}}, \\ v_n =<q_n, q_n>, \end{cases} \tag{5.12}$$

then the constrained Toda flow (5.11) is none other than the well-known Toda spectral problem

$$L\psi_n \equiv (E^{-1}u_n + v_n + u_n E)\psi_n = \lambda\psi_n, \quad Ef_n = f_{n+1}, \quad E^{-1}f_n = f_{n-1} \tag{5.13}$$

with the above constraint (5.12), $\lambda = \lambda_j$ and $\psi_n = q_{n,j}$. Theorem 5.2 shows that the constrained Toda flow (H_T) has the discrete Lax representation (5.10). Eq. (5.12) is a kind of discrete Bargmann constraint [44] of the Toda spectral problem (5.13).

The Hamiltonian systems (H_T) and (H_C) share the common Lax matrix (5.1). Thus, they have the following same r-matrix:

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda}P - S, \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.14}$$

which is proven to satisfy the classical Yang-Baxter equation (YBE)

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{kj}, r_{ik}] = 0, \ i, j, k = 1, 2, 3. \tag{5.15}$$

5.2 The restricted Toda and CKdV flows

Let us now consider the case on a symplectic manifold. We restrict the Toda and CKdV flows on the following symplectic submanifold M in R^{2N}

$$M = \{(q, p) \in R^{2N} | F \equiv <q, q> -1 = 0, G \equiv <q, p> -\frac{1}{2} = 0\}. \tag{5.16}$$

Let us first introduce the Dirac bracket

$$\{f, g\}_D = \{f, g\} + \frac{1}{2}(\{f, F\}\{G, g\} - \{f, G\}\{F, g\}) \tag{5.17}$$

which is easily proven to be a Poisson bracket on M.

According to the thought of ref. [36], the following Lax matrix

$$L_R^{TC} = L_R^{TC}(\lambda, p, q) = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + L_0 \equiv \begin{pmatrix} A_R(\lambda) & B_R(\lambda) \\ C_R(\lambda) & -A_R(\lambda) \end{pmatrix} \tag{5.18}$$

yields

$$-\lambda^2 \det L_R^{TC} = \frac{1}{2}\lambda^2 Tr(L_R^{TC})^2 = \frac{1}{4}\lambda^2 + < p, q > \lambda + 2H_R^C + \sum_{j=1}^{N} \frac{\lambda_j^2 E_{R,j}^{TC}}{\lambda - \lambda_j} \quad (5.19)$$

where

$$\begin{aligned} H_R^C &= \frac{1}{2} < \Lambda p, q > -\frac{1}{2} < q, q >< p, p > +\frac{1}{2} < p, q >^2, &(5.20)\\ E_{R,j}^{TC} &= E_{R,j}^{TC}(p,q) = p_j q_j - \Gamma_j,\ i = 1, \ldots, N, &(5.21)\end{aligned}$$

and L_0 is defined by (5.2).

An important observation is: if we consider the **Hamiltonian canonical equation produced by (5.20) in $\mathbf{R^{2N}}$, then this equation is exactly the well-known constrained AKNS flow**, which will be discussed in the next subsection. Now, we first consider the **Hamiltonian canonical equation restricted on M**:

$$(H_R^C): \quad q_x = \{q, H_R^C\}_D,\ p_x = \{p, H_R^C\}_D, \quad (5.22)$$

which reads as the following finite dimensional system:

$$\begin{cases} p_x = -\frac{1}{2}\Lambda p + \frac{1}{2}(< \Lambda q, q > -1)p + < p, p > q, \\ q_x = -p + \frac{1}{2}\Lambda q - \frac{1}{2}(< \Lambda q, q > -1)q, \\ < q, q >= 1, < q, p >= \frac{1}{2}. \end{cases} \quad (5.23)$$

This is actually the CKdV spectral problem (5.7) with the two constraints (Neumann-type) [13]

$$u =< \Lambda q, q > -1, \quad v =< p, p > \quad (5.24)$$

and $\lambda = \lambda_j,\ \psi = (p_j, q_j),\ j = 1, 2, ..., N$. So, the *finite dimensional system (5.23) coincides with the restricted CKdV (r-CKdV) flow.*

Let us return to the Lax matrix (5.18). After endowing with an auxiliary matrix $M_{T,R}$ as follows

$$M_{T,R} = \begin{pmatrix} 0 & a \\ -\frac{1}{a} & \frac{\lambda - b}{a} \end{pmatrix}, \quad (5.25)$$

$$a^2 =< \Lambda q - p, \Lambda q - p > + < \Lambda q, q > - < \Lambda q, q >^2,$$

$$b =< \Lambda q, q > -1,$$

then we have the following theorem.

Theorem 5.2.1 *The discrete Lax equation*

$$(L_R^{TC})' M_{T,R} = M_{T,R} L_R^{TC}, \quad (L_R^{TC})' = L_R^{TC}(\lambda, p', q') \quad (5.26)$$

is equivalent to a discrete Neumann type of finite dimensional symplectic map $\mathcal{H}_{\mathcal{T}}$: $(p,q)^T \to (p',q')^T$

$$\begin{cases} p' = aq, \\ q' = a^{-1}(\Lambda q - p - bq), \\ <q,q>=1, \ <q,p>=\frac{1}{2}, \end{cases} \tag{5.27}$$

which is called the restricted Toda (r-Toda) flow.

Remark 5.2.1 If we understand the above two matrices $(L_R^{TC})'$ and $M_{T,R}$ in the following sense: $(L_R^{TC})' \to (L_R^{TC})_{n+1}$, $M_{T,R} \to (M_{T,R})_n$ (i.e., $q \to q_n, p \to p_n, a \to a_n, b \to b_n$, here n is the discrete variable), then the restricted Toda flow (5.27) on the symplectic submanifold $M = \{(q,p) \in R^{2N} | <q,q>=1, <q,p>=\frac{1}{2}\}$ is nothing but the discrete Neumann system studied by Ragnisco [43].

Let $L_{R1}^{TC} = L_R^{TC}(\lambda,p,q) \otimes I$ and $L_{R2}^{TC} = I \otimes L_R^{TC}(\mu,p,q)$. Then, under the Dirac bracket (5.17) we obtain the following theorem.

Theorem 5.2.2 *The Lax matrix $L_R^{TC}(\lambda,p,q)$ defined by (5.18) satisfies the following fundamental Dirac-Poisson bracket*

$$\{L_R^{TC}(\lambda) \overset{\otimes}{,} L_R^{TC}(\mu)\}_D = [r_{12}(\lambda,\mu), L_{R1}^{TC}(\lambda)] - [r_{21}(\mu,\lambda), L_{R2}^{TC}(\mu)] \tag{5.28}$$

with a dynamical r-matrix

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P - S_{12}(\lambda,\mu), \quad r_{21}(\mu,\lambda) = Pr_{12}(\mu,\lambda)P \tag{5.29}$$

where

$$S_{12} = (E_{11}-E_{22}) \otimes E_{12} + E_{11} \otimes \begin{pmatrix} C_R(\mu) & 0 \\ 0 & 0 \end{pmatrix} + E_{12} \otimes \begin{pmatrix} 0 & -B_R(\mu) \\ C_R(\mu) & 0 \end{pmatrix} + E_{22} \otimes \begin{pmatrix} 0 & 2A_R(\mu) \\ 0 & C_R(\mu) \end{pmatrix} \tag{5.30}$$

and $P = \frac{1}{2}(I + \sum_{j=1}^3 \sigma_j \otimes \sigma_j)$ is the permutation matrix, σ_j $(j=1,2,3)$ are the Pauli matrices.

This Theorem ensures that (5.21) satisfies

$$\{E_{R,i}^{TC}, E_{R,j}^{TC}\}_D = 0, \ i,j,=1,\ldots,N. \tag{5.31}$$

For the r-Toda flow (5.27), we have $E_{R,i}^{TC}(p',q') = E_{R,i}^{TC}(p,q)$ as well as $\sum_{i=1}^n E_{R,i}^{TC} =< p,q >= \frac{1}{2}$ from the discrete Lax equation (5.26). Thus, in the set $\left\{E_{R,j}^{TC}\right\}_{j=1}^N$, only $E_{R,1}^{TC}, \ E_{R,2}^{TC}, \ldots, \ E_{R,N-1}^{TC}$ are independent on M. Therefore we obtain the following Theorem.

Theorem 5.2.3 *The restricted Toda flow $\mathcal{H}_{\mathcal{T}}$ is completely integrable, and its independent and invariant (N-1)-involutive system is $\{E_{R,i}^{TC}\}_{i=1}^{N-1}$.*

For the restricted CKdV flow on M, we have

$$H_R^C = \frac{1}{2}\sum_{j=1}^N \lambda_j E_{R,j}^{TC} \tag{5.32}$$

which implies

$$\{H_R^C, E_{R,j}^{TC}\}_D = 0, \quad j = 1, 2, \ldots, N. \tag{5.33}$$

Thus, the following Theorem holds.

Theorem 5.2.4 *The r-CKdV flow* (H_R^C) *is completely integrable, and its independent (N-1)-involutive system is also* $\{E_{R,k}^{TC}\}_{k=1}^{N-1}$.

Remark 5.2.3 As shown in this and last subsection, the r-Toda (i.e. Neumann-type) and the r-CKdV flows, and the c-Toda (i.e. Bargmann-type) and the c-CKdV flows respectively share the completely same Lax matrix, r-matrix and involutive conserved integrals. Thus, we say that the finite dimensional integrable CKdV flow both restricted and constrained is the interpolating Hamiltonian flow of invariant of the corresponding Toda integrable symplectic map.

5.3 The constrained AKNS and Dirac (D) flows

From now on we assume:

$$L_0 = L_0(\lambda, p, q) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} p_j q_j & -q_j^2 \\ p_j^2 & -p_j q_j \end{pmatrix}. \tag{5.34}$$

Let us again consider Eq. (5.18), and rewrite it as the following version:

$$L^{AKNS} = L^{AKNS}(\lambda, p, q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + L_0, \tag{5.35}$$

while we introduce

$$L^{D} = L^{D}(\lambda, p, q) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + L_0. \tag{5.36}$$

Then we have

$$\frac{1}{2}\lambda^2 Tr(L^{AKNS})^2(\lambda) = \lambda^2 + 2\lambda < p, q > + < p, q >^2 + 2H_{AKNS} + \sum_{j=1}^{N} \frac{\lambda_j^2 E_j^{AKNS}}{\lambda - \lambda_j}, \tag{5.37}$$

$$\frac{1}{2}\lambda^2 Tr(L^{D})^2(\lambda) = \lambda^2 + \lambda(< q, q > + < p, p >) - \frac{1}{4}(< p, p > + < q, q >)^2 + 2H_D + \sum_{j=1}^{N} \frac{\lambda_j^2 E_j^{D}}{\lambda - \lambda_j}, \tag{5.38}$$

where

$$H_{AKNS} = < \Lambda p, q > - \frac{1}{2} < q, q >< p, p >, \tag{5.39}$$

$$\begin{aligned} H_D = {} & \frac{1}{2}(< \Lambda q, q > + < \Lambda p, p >) + \frac{1}{2}(< p, q >^2 - < q, q >< p, p >) \\ & + \frac{1}{8}(< p, p > + < q, q >)^2, \end{aligned} \tag{5.40}$$

$$E_j^{AKNS} = 2p_j q_j - \Gamma_j, \ j = 1, \ldots, N, \tag{5.41}$$

$$E_j^{D} = p_j^2 + q_j^2 - \Gamma_j, \ j = 1, \ldots, N. \tag{5.42}$$

Thus, H_{AKNS} and H_D generate the following two Hamiltonian systems

$$(H_{AKNS}) \quad : \quad \begin{cases} q_x = \frac{\partial H_{AKNS}}{\partial p} = -<q,q>p + \Lambda q, \\ p_x = -\frac{\partial H_{AKNS}}{\partial q} = <p,p>q - \Lambda p; \end{cases} \tag{5.43}$$

$$(H_D) \quad : \quad \begin{cases} q_x = \frac{\partial H_D}{\partial p} = <p,q>q + \frac{1}{2}(<p,p> - <q,q>)p + \Lambda p, \\ p_x = -\frac{\partial H_D}{\partial q} = -<p,q>p - \frac{1}{2}(<p,p> - <q,q>)q - \Lambda q. \end{cases} \tag{5.44}$$

It can be easily seen that (H_{AKNS}) and (H_D) are changed to the well-known Zakharov-Shabat-AKNS spectral problem [54]

$$y_x = \begin{pmatrix} \lambda & u \\ v & -\lambda \end{pmatrix} y \tag{5.45}$$

and the Dirac spectral problem [30]

$$y_x = \begin{pmatrix} -v & \lambda - u \\ -\lambda - u & v \end{pmatrix} y \tag{5.46}$$

with the constraints $u = -<q,q>,\ v = <p,p>,\ \lambda = \lambda_j,\ y = (q_j, p_j)^T$, and the constraints $u = -\frac{1}{2}(<p,p> - <q,q>),\ v = -<p,q>,\ \lambda = \lambda_j,\ y = (q_j, p_j)^T$, respectively.

Therefore (H_{AKNS}) and (H_D) coincide with the constrained AKNS (c-AKNS) system and the constrained Dirac (c-D) system, respectively.

Let $L_1^J(\lambda) = L^J(\lambda) \otimes I$, $L_2^J(\mu) = I \otimes L^J(\mu)$ $(J = AKNS, D)$. Then we have the following theorem.

Theorem 5.3.1 *The Lax matrices $L^J(\lambda)$ $(J = AKNS, D)$ defined by Eq. (5.35) and Eq. (5.36) satisfy the fundamental Poisson bracket*

$$\{L^J(\lambda) \overset{\otimes}{,} L^J(\mu)\} = [r_{12}(\lambda,\mu), L_1^J(\lambda)] - [r_{21}(\mu,\lambda), L_2^J(\mu)]. \tag{5.47}$$

Here the r-matrices $r_{12}(\lambda,\mu)$, $r_{21}(\mu,\lambda)$ are exactly given by the following standard r-matrix

$$r_{12}(\lambda,\mu) = \frac{2}{\mu - \lambda} P, \quad r_{21}(\mu,\lambda) = P r_{12}(\mu,\lambda) P, \tag{5.48}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}(I + \sum_{i=1}^{3} \sigma_i \otimes \sigma_i). \tag{5.49}$$

So, the c-AKNS and c-D flows share the same standard r-matrix (5.48), which is obviously non-dynamical. However, **the two constrained flows, produced by**

(5.45)'s and (5.46)'s extensive spectral problems (6.1) and (6.2) (they are gauge equivalent), have different r-matrices (see section 6).

Remark 5.3.1 In fact, the r-matrix $r_{12}(\lambda,\mu)$ in the case of the c-AKNS and c-D flows can be also chosen as

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P + I\otimes\tilde{S},\ \ \tilde{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.50}$$

where the elements a, b, c, d can be arbitrary C^∞-functions $a(\lambda,\mu,p,q), b(\lambda,\mu,p,q), c(\lambda,\mu,p,q), d(\lambda,\mu,p,q)$ with respect to the spectral parametres λ,μ and the dynamical variables p, q. This shows that for a given Lax matrix, the associated r-matrix is not uniquely defined (there are even infinitely many r-matrices possible). Here we give the simplest case: $a = b = c = d = 0$, i. e. the standard r-matrix (5.48).

5.4 The constrained Harry-Dym (HD) and Heisenberg spin chain (HSC) flows

The constrained Harry-Dym system describes the geodesic flow on an ellipsoid and shares the same r-matrix with the constrained Heisenberg spin chain (HSC). To prove this, we consider the following Lax matrices:

$$L^{HD} = L^{HD}(\lambda,p,q) = \begin{pmatrix} -<p,q>\lambda^{-1} & \lambda^{-2}+<q,q>\lambda^{-1} \\ -<p,p>\lambda^{-1} & <p,q>\lambda^{-1} \end{pmatrix} + L_0, \tag{5.51}$$

$$L^{HSC} = L^{HSC}(\lambda,p,q) = \begin{pmatrix} -<p,q>\lambda^{-1} & <q,q>\lambda^{-1} \\ -<p,p>\lambda^{-1} & <p,q>\lambda^{-1} \end{pmatrix} + L_0. \tag{5.52}$$

Here L^{HSC} is included in the generalized Lax matrix (2.1), but L^{HD} is not. We need two associated auxiliary matrices

$$M_{HD} = \begin{pmatrix} 0 & 1 \\ -\frac{<\Lambda p,p>}{<\Lambda^2 q,q>}\lambda & 0 \end{pmatrix}, \tag{5.53}$$

$$M_{HSC} = \begin{pmatrix} -i\lambda<\Lambda p,q> & i\lambda<\Lambda q,q> \\ -i\lambda<\Lambda p,p> & i\lambda<\Lambda p,q> \end{pmatrix}, \quad i^2=-1. \tag{5.54}$$

Theorem 5.4.1 *The Lax representations*

$$L_x^{HD} = [M_{HD}, L^{HD}], \tag{5.55}$$

$$L_x^{HSC} = [M_{HSC}, L^{HSC}] \tag{5.56}$$

respectively give the following fininte dimensional Hamiltonian flows:

$$(H_{HD}): \begin{cases} q_x = p = \frac{\partial H_{HD}}{\partial p}|_{TQ^{N-1}}, \\ p_x = -\frac{<\Lambda p,p>}{<\Lambda^2 q,q>}\Lambda q = -\frac{\partial H_{HD}}{\partial q}|_{TQ^{N-1}}, \\ <\Lambda q,q>=1; \end{cases} \tag{5.57}$$

$$(H_{HSC}): \begin{cases} q_x = i < \Lambda q, q > \Lambda p - i < \Lambda p, q > \Lambda q = \frac{\partial H_{HSC}}{\partial p}, \\ p_x = i < \Lambda p, q > \Lambda p - i < \Lambda p, p > \Lambda q = -\frac{\partial H_{HSC}}{\partial q}, \end{cases} \tag{5.58}$$

with the Hamiltonian functions

$$H_{HD} = \frac{1}{2} < p, p > - \frac{< \Lambda p, p >}{2 < \Lambda^2 q, q >}(< \Lambda q, q > -1), \tag{5.59}$$

$$H_{HSC} = \frac{1}{2} i < \Lambda p, p >< \Lambda q, q > - \frac{1}{2} i < \Lambda p, q >^2 . \tag{5.60}$$

In Eq. (5.57) TQ^{N-1} is a tangent bundle in R^{2N}:

$$TQ^{N-1} = \{(p, q) \in R^{2N} | F \equiv < \Lambda q, q > -1 = 0, G \equiv < \Lambda p, q >= 0\}. \tag{5.61}$$

Obviously, Eq. (5.57) is equivalent to

$$q_{xx} + \frac{< \Lambda q_x, q_x >}{< \Lambda^2 q, q >} \Lambda q = 0, \ < \Lambda q, q >= 1, \tag{5.62}$$

which is nothing but the equation of the geodesic flow [26] on the surface $< \Lambda q, q >= 1$ in the space R^N and also coincides with the constrained HD (c-HD) flow [9]. In addition, Eq. (5.58) becomes the Heisenberg spin chain spectral problem [49]

$$y_x = \begin{pmatrix} -i\lambda w & -i\lambda u \\ -i\lambda v & i\lambda w \end{pmatrix} y, \quad i^2 = -1, \tag{5.63}$$

with the constraints $u = - < \Lambda q, q >$, $v =< \Lambda p, p >$, $w = - < \Lambda p, q >$, $\lambda = \lambda_j$, $y = (q_j, p_j)^T$. Thus, Eq. (5.58) reads as the constrained Heisenberg spin chain (c-HSC) flow [40].

Their Lax matrices (5.51) and (5.52) share all elements except one, namely

$$\begin{pmatrix} 0 & \lambda^{-2} \\ 0 & 0 \end{pmatrix}.$$

This element does not affect the calculations concerning the fundamental Poisson bracket, one can readily deduce that the c-HD flow and the c-HSC flow possess the same non-dynamical r-matrix

$$r_{12}(\lambda, \mu) = \frac{2\lambda}{\mu(\mu - \lambda)} P, \ r_{21}(\mu, \lambda) = P r_{12}(\mu, \lambda) P. \tag{5.64}$$

Remark 5.4.1 The r-matrix (5.64) of the c-HD and c-HSC flows can be also chosen as

$$r_{12}(\lambda, \mu) = \frac{2\lambda}{\mu(\mu - \lambda)} P + I \otimes \tilde{S}. \tag{5.65}$$

Evidently, Eq. (5.64) is the simplest case: $\tilde{S} = 0$ of Eq. (5.65).

5.5 The constrained G and Q flows

In this subsection, we introduce the following Lax matrices:

$$L^G = L^G(\lambda, p, q) = \begin{pmatrix} (\frac{1}{2}+ < p, q >)\lambda^{-1} & < q, q > \lambda^{-1} \\ 0 & -(\frac{1}{2}+ < p, q >)\lambda^{-1} \end{pmatrix} + L_0, \quad (5.66)$$

$$L^Q = L^Q(\lambda, p, q) = \begin{pmatrix} -\lambda^{-1} & < q, q > \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix} + L_0. \quad (5.67)$$

If we set

$$M_G = \begin{pmatrix} -\frac{1}{\alpha}\lambda & \frac{1}{\alpha}(< p, p > - < q, q >) - 1 \\ \frac{1}{\alpha}(< p, p > - < q, q > +1)\lambda & \frac{1}{\alpha}\lambda \end{pmatrix}, \quad (5.68)$$

$$M_Q = \begin{pmatrix} \lambda + \frac{1}{2\beta^2} < \Lambda q, q >< p, p > & \frac{1}{\beta} < \Lambda q, q > \\ -\frac{1}{\beta} < p, p > \lambda & -\lambda - \frac{1}{2\beta^2} < \Lambda q, q >< p, p > \end{pmatrix}, \quad (5.69)$$

with

$$\alpha = \sqrt{(< p, p > - < \Lambda q, q >)^2 - 4 < \Lambda q, p >}, \quad \beta = 1- < p, q >, \quad (5.70)$$

then, by a lengthy and straightforward calculation we obtain the following theorem.

Theorem 5.5.1 *The following Lax representations*

$$L_x^G = [M_G, L^G] \quad (5.71)$$

and

$$L_x^Q = [M_Q, L^Q] \quad (5.72)$$

where the first one is restricted to the surface $M_1 = \{(p, q) \in R^{2N} | < p, q >= 0, < \Lambda q, q >< p, p > + < \Lambda q, p >= 0\}$ *in the space* R^{2N}*, respectively produce the finite-dimensional systems:*

$$\begin{cases} q_x = \frac{1}{\alpha}(-\Lambda q + (< p, p > - < \Lambda q, q >)p) - p, \\ p_x = \frac{1}{\alpha}(\Lambda p + (< p, p > - < \Lambda q, q >)\Lambda q) + \Lambda q, \end{cases} \quad (5.73)$$

and

$$\begin{cases} q_x = \Lambda q + \frac{1}{\beta} < \Lambda q, q > p + \frac{1}{2\beta^2} < p, p >< \Lambda q, q > q, \\ p_x = -\Lambda p - \frac{1}{\beta} < p, p > \Lambda q - \frac{1}{2\beta^2} < p, p >< \Lambda q, q > p. \end{cases} \quad (5.74)$$

Eqs. (5.73) and (5.74) turn out to be the spectral problem studied by Geng (simply called G-spectral problem) [22]

$$y_x = \begin{pmatrix} -\lambda u & v - 1 \\ \lambda(v + 1) & \lambda u \end{pmatrix} y \quad (5.75)$$

with the constraint condition

$$\begin{aligned} u &= \frac{1}{\alpha} = \frac{1}{\sqrt{(<p,p> - <\Lambda q,q>)^2 - 4<\Lambda q,p>}}, \\ v &= \frac{<p,p> - <\Lambda q,q>}{\alpha} = \frac{<p,p> - <\Lambda q,q>}{\sqrt{(<p,p> - <\Lambda q,q>)^2 - 4<\Lambda q,p>}}, \end{aligned}$$

$\lambda = \lambda_j,\ y = (q_j, p_j)^T$, and the spectral problem proposed by Qiao (simply called Q-spectral problem) [39]

$$y_x = \begin{pmatrix} \lambda - \frac{1}{2}uv & u \\ \lambda v & -\lambda + \frac{1}{2}uv \end{pmatrix} y \tag{5.76}$$

with the constraint condition

$$\begin{aligned} u &= \frac{<\Lambda q,q>}{\beta} = \frac{<\Lambda q,q>}{1 - <q,p>}, \\ v &= \frac{-<p,p>}{\beta} = -\frac{<p,p>}{1 - <q,p>}, \end{aligned}$$

$\lambda = \lambda_j,\ y = (q_j, p_j)^T$, respectively.

So, Eqs. (5.73) and (5.74) are the constrained Geng (c-G) flow and the constrained Qiao (c-Q) flow, and they have the same non-dynamical r-matrix:

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P - \frac{2}{\mu}S,\ S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \sigma_- \otimes \sigma^+. \tag{5.77}$$

Here, the r-matrix $r_{12}(\lambda,\mu)$ can be also chosen as

$$r_{12}(\lambda,\mu) = \frac{2}{\mu-\lambda}P - \frac{2}{\mu}S + I \otimes \tilde{S}. \tag{5.78}$$

Eq. (5.77) is the simplest case: $\tilde{S} = 0$ of Eq. (5.78).

We have already seen that the r-matrix $r_{12}(\lambda,\mu)$ satisfying the fundamental Poisson bracket is not unique (in fact, infinitely many) and is usually composed of two parts, the first one being their main term, and the second one being the common term $I \otimes \tilde{S}$. Usually, to prove the integrability we choose their main term as the simplest r-matrix.

6 An equivalent pair with different r-matrices

This section reveals the following interesting fact: a pair of constrained systems, produced by two gauge equivalent spectral problems, possesses different r-matrices.

In 1992, Geng introduced the following spectral problem [23]

$$\phi_x = M\phi,\ M = \begin{pmatrix} i\lambda - i\beta uv & u \\ v & -i\lambda + i\beta uv \end{pmatrix},\ i^2 = -1 \tag{6.1}$$

where u and v are two scalar potentials, λ is a spectral parameter and β is a constant, and discussed its evolution equations and Hamiltonian structure. Eq. (6.1) is apparently an extension of the AKNS spectral problem (5.45). Two years later the author considered an extension of the Dirac spectral problem (5.46) [41]

$$\psi_x = \overline{M}\psi,\ \overline{M} = \begin{pmatrix} -is & \lambda + r + \beta(s^2 - r^2) \\ -\lambda + r - \beta(s^2 - r^2) & is \end{pmatrix}, \tag{6.2}$$

where $r,\ s$ are two potentials, and obtained a finite dimensional involutive system being not equivalent to that one in ref. [23]. But, the spectral problems (6.1) and (6.2) are gauge equivalent via the following transformation [52]

$$\psi = G\phi,\ G = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{6.3}$$

$v = i(r - s),\ u = -i(r + s)$. In ref. [52], Wadati and Sogo discussed the gauge transformations of some spectral problems like Eq. 5.63.

Now, we discuss their r-matrices. Let us consider the following two Lax matrices:

$$L^{GX} = L^{GX}(\lambda, p, q) = \begin{pmatrix} 1 + 2i\beta < p, q > & 0 \\ 0 & -1 - 2i\beta < p, q > \end{pmatrix} - iL_0, \tag{6.4}$$

$$L^{QZ} = L^{QZ}(\lambda, p, q) = \begin{pmatrix} 0 & \frac{1}{2} - \beta(< p, p > + < q, q >) \\ -\frac{1}{2} + \beta(< p, p > + < q, q >) & 0 \end{pmatrix} + L_0. \tag{6.5}$$

Then calculating their determinants leads to the following Hamiltonian systems

$$\begin{cases} q_x = \frac{\partial H_{GX}}{\partial p} = \Lambda q + i\beta \frac{<p,p><q,q>}{(1+2i\beta<p,q>)^2} q - \frac{<q,q>}{1+2i\beta} p, \\ p_x = -\frac{\partial H_{GX}}{\partial q} = -\Lambda p - i\beta \frac{<p,p><q,q>}{(1+2i\beta<p,q>)^2} p + \frac{<p,p>}{1+2i\beta} q, \end{cases} \tag{6.6}$$

and

$$\begin{cases} q_x = \frac{\partial H_{QZ}}{\partial p} = \Lambda p - \beta \frac{4<p,q>^2 + (<p,p> - <q,q>)^2}{(1-2\beta(<p,p>+<q,q>))^2} p - \frac{2<p,q>q + (<p,p> - <q,q>)p}{1-2\beta(<q,q>+<p,p>)}, \\ p_x = -\frac{\partial H_{QZ}}{\partial q} = -\Lambda q + \beta \frac{4<p,q>^2 + (<p,p> - <q,q>)^2}{(1-2\beta(<p,p>+<q,q>))^2} q + \frac{2<p,q>p - (<p,p> - <q,q>)q}{1-2\beta(<q,q>+<p,p>)}, \end{cases} \tag{6.7}$$

with the Hamiltonian functions

$$H_{GX} = i < \Lambda q, p > - \frac{< p, p >< q, q >}{2(1 + 2i\beta < p, q >)} \tag{6.8}$$

and

$$H_{QZ}=\frac{1}{2}<\Lambda p,p>+\frac{1}{2}<\Lambda q,q>-\frac{4<p,q>^2+(<p,p>-<q,q>)^2}{4-8\beta(<p,p>+<q,q>)}. \qquad (6.9)$$

Obviously, Eqs. (6.6) and (6.7) become Eqs. (6.1) and (6.2) with the constrants

$$u=-\frac{<q,q>}{1+2i\beta<p,q>},\ v=\frac{<p,p>}{1+2i\beta<p,q>}, \qquad (6.10)$$

$\lambda=\lambda_j,\ \phi=(q_j,p_j)^T,\ j=1,\dots,N$; and the constraints

$$s=\frac{-2i<p,q>}{1-2\beta(<q,q>+<p,p>)},\ r=\frac{-<p,p>+<q,q>}{1-2\beta(<q,q>+<p,p>)}, \qquad (6.11)$$

$\lambda=\lambda_j,\ \psi=(q_j,p_j)^T,\ j=1,\dots,N$, respectively. Thus, the finite dimensional Hamiltonian systems (6.6) and (6.7) are respectively the constrained flows of the spectral problems (6.1) and (6.2). Since they have Lax matrices (6.4) and (6.5), then the r-matrices of (6.6) and (6.7) are respectively:

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+4i\beta S,\ S=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}, \qquad (6.12)$$

and

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+2\beta S,\ S=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}. \qquad (6.13)$$

which are apparently different.

7 New integrable systems

In this section, three new integrable systems are generated as the representatives from our generalized r-matrix structure.

1. The first system is given by case 6 in section 4. The corresponding r-matrix and involutive systems are respectively

$$r_{12}(\lambda,\mu)=\frac{2}{\mu-\lambda}P+\frac{2}{\mu}S,\ S=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix} \qquad (7.1)$$

and

$$E_j^1 = 2(< p, q > +c)\lambda_j^{-1} p_j q_j + < q, q > \lambda_j^{-1} p_j^2 - \Gamma_j,\ j = 1, \ldots, N, \tag{7.2}$$

where $c \in R$. Thus, the finite dimesional Hamiltonian systems (F_m^1) defined by $F_m^1 = \sum_{j=1}^N \lambda_j^m E_j^1$, $m = 0, \ldots$, i.e.

$$\begin{aligned} F_m^1 &= 2(< p, q > +c) < \lambda_j^{m-1} p, q > + < q, q >< \lambda_j^{m-1} p, p > \\ &- \sum_{i+j=m-1} (< \Lambda^i q, q >< \Lambda^j p, p > - < \Lambda^i q, p >< \Lambda^j p, q >) \end{aligned} \tag{7.3}$$

are completely integrable. Particularly, with $m = 2$ the Hamiltonian system (F_2^1):

$$\begin{cases} q_x = \frac{\partial F_2^1}{\partial p} = 2c\Lambda q - 2 < \Lambda q, q > p + 4 < \Lambda p, q > q + 4 < p, q > \Lambda q, \\ p_x = -\frac{\partial F_2^1}{\partial q} = -2c\Lambda p + 2 < p, p > \Lambda q - 4 < \Lambda p, q > p - 4 < p, q > \Lambda p, \end{cases} \tag{7.4}$$

is a new integrable system, which becomes the following spectral problem

$$\phi_x = \begin{pmatrix} (2c + 4v)\lambda + 4u & -2w \\ 2s\lambda & -(2c + 4v)\lambda - 4u \end{pmatrix} \phi \tag{7.5}$$

with the constraint conditions $u =< \Lambda p, q >,\ v =< p, q >,\ w =< \Lambda q, q >,\ s =< p, p >$, and $\lambda = \lambda_j,\ \phi = (q_j, p_j)^T,\ j = 1, \ldots, N$. Apparently, the spectral problem (7.5) is new.

2. The second system is produced by case (7.1) in section 4. The corresponding r-matrix and involutive systems are respectively

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P + \frac{2}{\mu} S,\ S = \begin{pmatrix} 0 & g_2' & 0 & 0 \\ f_2' & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.6}$$

and

$$E_j^2 = 2c\lambda_j^{-1} p_j q_j + (< q, q > +g_2)\lambda_j^{-1} p_j^2 + (< p, p > -f_2)\lambda_j^{-1} q_j^2 - \Gamma_j,\ j = 1, \ldots, N \tag{7.7}$$

where $c \in R,\ f_2 = f_2(< p, q >),\ g_2 = g_2(< p, q >) \in C^\infty(R)$. Hence, the Hamiltonian system (F_2^2) defined by $F_2^2 = \sum_{j=1}^N \lambda_j^2 E_j^2$, i.e.

$$F_2^2 = 2c < \Lambda p, q > +2 < \Lambda p, q >< p, q > +g(< p, q >) < \Lambda p, p > -f(< p, q >) < \Lambda q, q > \tag{7.8}$$

is completely integrable. Meanwhile the Hamiltonian system (F_2^2):

$$\begin{aligned} q_x = \frac{\partial F_2^2}{\partial p} &= 2c\Lambda q + 2 < \Lambda q, p > q + 2 < p, q > \Lambda q \\ &+ 2g(< p, q >)\Lambda p + < \Lambda p, p > g'(< p, q >)q - < \Lambda q, q > f'(< p, q >)q, \end{aligned} \tag{7.9}$$

$$\begin{aligned} p_x = -\frac{\partial F_2^2}{\partial q} &= -2c\Lambda p - 2 < \Lambda q, p > p - 2 < p, q > \Lambda p \\ &- < \Lambda p, p > g'(< p, q >)p + 2f(< p, q >)\Lambda q - < \Lambda q, q > f'(< p, q >)p, \end{aligned} \tag{7.10}$$

can also related to a new 2×2 spectral problem with some constraint conditions.

3. The third system is derived by case (7.4) in section 4. The corresponding r-matrix and involutive systems are respectively

$$r_{12}(\lambda, \mu) = \frac{2}{\mu - \lambda} P - \frac{2}{\mu} S, \;\; S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c'_{-1} & 1 \end{pmatrix} \tag{7.11}$$

and

$$E_j^3 = 2(- < p, q > + c)\lambda_j^{-1} p_j q_j + < q, q > \lambda_j^{-1} p_j^2 - c_{-1} \lambda_j^{-1} q_j^2 - \Gamma_j \tag{7.12}$$

where $c \in R$ and $c_{-1} = c_{-1}(< p, q >) \in C^{\infty}(R)$. The following Hamiltonian system (F_2^3):

$$\begin{cases} q_x = \frac{\partial F_2^3}{\partial p} = 2c\Lambda q - < \Lambda q, q > (c'_{-1}(< p, q >) q + 2p), \\ p_x = -\frac{\partial F_2^3}{\partial q} = -2c\Lambda p + 2(c_{-1} + < p, p >)\Lambda q + c_{-1}(< p, q >) < \Lambda q, q > p, \end{cases} \tag{7.13}$$

is one of their products, where

$$F_2^3 \;\; = \;\; 2c < \Lambda p, q > - < \Lambda q, q > (c_{-1} + < p, p >). \tag{7.14}$$

In general, with any c_{-1} Eq. (7.13) can't be changed to a 2×2 spectral problem with some constraints. But with two special c_{-1}: $c_{-1} = 0$ and $c_{-1} =< p, q >$, Eq. (7.13) can respectively become the spectral problem [31]

$$\phi_x = \begin{pmatrix} 2c\lambda & -2v \\ 2u\lambda & -2c\lambda \end{pmatrix} \phi \tag{7.15}$$

with the constraint conditions $u =< p, p >, \; v =< \Lambda q, q >$, and the spectral problem

$$\phi_x = \begin{pmatrix} 2c\lambda - v & -2v \\ 2u\lambda & -2c\lambda + v \end{pmatrix} \phi \tag{7.16}$$

with the constraint conditions $u =< p, q + p >, \; v =< \Lambda q, q >$. Here in Eqs. (7.15) and (7.16) $\lambda = \lambda_j, \; \phi = (q_j, p_j)^T, \; j = 1, \ldots, N$ are set. Eq. (7.16) is a new spectral problem.

Remark 7.1 We can consider further new integrable systems induced by Theorem 3.1.

Remark 7.2 The above procedure actually give an approach how to connect an r-matrix of finite dimensional system with a spectral problem, which is closely associated with integrable NLEEs.

8 Algebro-geometric solutions

The ideal aim for nonlinear differential equations is of course to obtain their explicit solution. In this section, we connect the finite dimesional integrable flows with integrable NLEEs, and solve them with an explicit form of algebro-geometric solutions. Here, we take two examples: one being the periodic or infinite Toda lattice equation, the other the AKNS equation with the condition of decay at infinity or periodic boundary.

8.1 Toda lattice equation

The Toda hierarchy associated with Eq. (5.13) is derived as follows:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_j} = JG_j^n, \quad j = 0, 1, 2, \dots \tag{8.1}$$

where $\{G_j^n = J^{-1}KG_{j-1}^n\}_{j=0}^{\infty}$ is the Lenard sequence, $G_{-1}^n = (\alpha u_n^{-1}, \beta)^T \in KerJ$, for all $\alpha = \alpha(t_j),\ \beta = \beta(t_j) \in C^\infty(R)$, the two symmetric operators K, J are

$$K = \begin{pmatrix} \frac{1}{2}u_n(E - E^{-1})u_n & u_n(E-1)v_n \\ v_n(1 - E^{-1})u_n & 2(u_n^2 E - E^{-1}u_n^2) \end{pmatrix},\ J = \begin{pmatrix} 0 & u_n(E-1) \\ (1 - E^{-1})u_n & 0 \end{pmatrix}. \tag{8.2}$$

In particular, with $j = 0$, $\beta = 1$ Eq. (8.1) reads as the Toda lattice

$$\dot{u}_n = u_n(v_{n+1} - v_n), \quad \dot{v}_n = 2(u_n^2 - u_{n-1}^2) \tag{8.3}$$

which can be changed to

$$\ddot{x}_n = 2(e^{2(x_{n+1}-x_n)} - e^{2(x_n - x_{n-1})}) \tag{8.4}$$

via the following transformation

$$u_n = e^{x_{n+1}-x_n}, \quad v_n = \dot{x}_n. \tag{8.5}$$

It is easy to prove the following theorem.

Theorem 8.1.1

1) Let $\hat{G}_n = (\hat{G}_n^{(1)}, \hat{G}_n^{(2)})^T,\ \forall \hat{G}_n^{(1)}, \hat{G}_n^{(2)} \in C^\infty(R)$. Then the operator equation

$$\Big[V(\hat{G}_n), L\Big] = L_*(K\hat{G}_n) - L_*(J\hat{G}_n)L$$

possesses the operator solution

$$V(\hat{G}_n) = -(E^{-1}u_n)\hat{G}_n^{(2)}E^{-1} + \frac{1}{2}((E^{-1}u_n\hat{G}_n^{(1)}) - u_n\hat{G}_n^{(1)}) + u_n\hat{G}_n^{(2)}E \tag{8.6}$$

where $[\cdot,\cdot]$ is the usual commutator; the operator L is defined by Eq. (5.13); $L_*(\xi) = E^{-1}\xi_1 + \xi_2 + \xi_1 E,\ \forall \xi = (\xi_1, \xi_2)^T, \xi_1, \xi_2 \in C^\infty(R)$.

2) Let us choose the special $\hat{G}_n = G_j^n,\ j = -1, 0, 1, ...,$ then the Toda hierarchy (8.1) has the following Lax representation of operator form

$$L_{t_j} = [W(G_j^n), L], \quad j = 0, 1, 2... \tag{8.7}$$

where the operator $W(G_j^n) = \sum_{k=0}^{j} V(G_{k-1}^n) L^{j-k}$.

Particularly, the standard Toda equation (8.4) possesses the Lax representation of operator form $L_t = [W(G_0^n), L]$, where the operator $W(G_0^n) = e^{x_{n+1}-x_n}E - e^{x_n - x_{n-1}}E^{-1}$, and $u_n,\ v_n$ in L are substituted by (8.5).

We have shown that the c-Toda flow and the c-CKdV flow share a common nondynamical r-matrix and in particular, this ensures the integrability of their flows. A calculation of determinant yields their common N-involutive systems

$$E_\alpha = \lambda_\alpha p_\alpha q_\alpha - p_\alpha^2 - < p, q > q_\alpha^2 - \sum_{\beta \neq \alpha, \beta = 1}^{N} \frac{(q_\alpha p_\beta - p_\alpha q_\beta)^2}{\lambda_\alpha - \lambda_\beta}, \alpha = 1, \dots, N \tag{8.8}$$

which are independent and invariant (i.e. $E_\alpha(\lambda, p, q) = E_\alpha(\lambda, p', q')$). Apparently, the functions $F_s = \sum_{\alpha=1}^{N} \lambda_\alpha^s E_\alpha,\ s = 0, 1, 2...,$ are given by

$$\begin{aligned} F_s &= < \Lambda^{s+1} p, q > - < \Lambda^s p, p > - < p, q >< \Lambda^s q, q > \\ &\quad - \sum_{j+k=s-1} (< \Lambda^j p, p >< \Lambda^k q, q > - < \Lambda^j p, q >< \Lambda^k q, p >) \end{aligned} \tag{8.9}$$

and $\{F_m, F_l\} = 0,\ \forall m, l \in Z^+$ which implies the Hamiltonian systems (F_s) are completely integrable.

Let $(p_0(t_s), q_0(t_s))^T$ be a solution of the initial problem

$$\frac{\partial}{\partial t_s} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\partial F_s/\partial q \\ \partial F_s/\partial p \end{pmatrix}, \quad \begin{pmatrix} p \\ q \end{pmatrix}_{t_s=0} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}. \tag{8.10}$$

Set

$$\begin{pmatrix} p_n(t_s) \\ q_n(t_s) \end{pmatrix} = H_T^n \begin{pmatrix} p_0(t_s) \\ q_0(t_s) \end{pmatrix} \tag{8.11}$$

where H_T is defined by Eq. (5.11). Now, we rewrite Eq. (5.12) as a map $f : R^{2N} \longrightarrow R^2$ defined by

$$f: \quad (p_n, q_n)^T \longmapsto (u_n, v_n)^T. \tag{8.12}$$

Then, we have the following theorem.

Theorem 8.1.2 *$(u_n(t_s), v_n(t_s))^T = f(p_n(t_s), q_n(t_s))$ satisfies the Toda hierarchy*

$$\frac{d}{dt_s} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = J G_s^n, \quad s = 0, 1, \dots. \tag{8.13}$$

Particularly, with $s=0$ the following calculable method

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \xrightarrow{F_0} \begin{pmatrix} p_0(t) \\ q_0(t) \end{pmatrix} \xrightarrow{H^n} \begin{pmatrix} p_n(t) \\ q_n(t) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} u_n(t) \\ v_n(t) \end{pmatrix} \tag{8.14}$$

produces a solution of Toda lattice equation (8.3). Thus, the standard Toda equation (8.4) has the following formal solution

$$x_n(t) = \int < q_n(t), q_n(t) > dt. \tag{8.15}$$

We shall concretely give the expression $< q_n(t), q_n(t) >$.

Let us rewrite the element $C_{TC}(\lambda)$ of Eq. (5.1) as

$$C_{TC}(\lambda) \equiv -\frac{Q(\lambda)}{K(\lambda)}, \quad K(\lambda) = \prod_{\alpha=1}^{N} (\lambda - \lambda_\alpha), \tag{8.16}$$

and choose N distinct real zero points $\mu_1, \ldots, \mu_N$ of $Q(\lambda)$. Then, we have

$$Q(\lambda) = \prod_{j=1}^{N} (\lambda - \mu_j), \ < q, q >= \sum_{\alpha=1}^{N} \lambda_\alpha - \sum_{j=1}^{N} \mu_j. \tag{8.17}$$

Let

$$\pi_j = A_{TC}(\mu_j), \tag{8.18}$$

then it is easy to prove the following proposition.

Proposition 8.1.1

$$\{\mu_i, \mu_j\} = \{\pi_i, \pi_j\} = 0, \quad \{\pi_j, \mu_i\} = \delta_{ij}, \quad i, j = 1, 2, ..., N, \tag{8.19}$$

i.e. π_j, μ_j are conjugated, and thus they are the seperated variables [48].

Write $\det L^{TC}(\lambda) = -A_{TC}^2(\lambda) - B_{TC}(\lambda)C_{TC}(\lambda) = -\frac{1}{4}\lambda^2 - \sum_{\alpha=1}^{N} \frac{E_\alpha}{\lambda - \lambda_\alpha} = -\frac{P(\lambda)}{K(\lambda)}$, where E_α is defined by (8.8), and $P(\lambda)$ is an $N+2$ order polynomial of λ whose first term's coefficient is $\frac{1}{4}$, then $\pi_j^2 = \frac{P(\mu_j)}{K(\mu_j)}$, $j = 1, \ldots, N$. Now, we choose the generating function

$$W = \sum_{j=1}^{N} W_j(\mu_j, \{E_\alpha\}_{\alpha=1}^{N}) = \sum_{j=1}^{N} \int_{\mu_j(0)}^{\mu_j(n)} \sqrt{\frac{P(\lambda)}{K(\lambda)}} d\lambda \tag{8.20}$$

where $\mu_j(0)$ is an arbitrary given constant. Let us view E_α ($\alpha = 1, \ldots, N$) as actional variables, then angle-coordinates Q_α are chosen as

$$Q_\alpha = \frac{\partial W}{\partial E_\alpha}, \quad \alpha = 1, \ldots, N$$

i.e.

$$Q_\alpha = \sum_{k=1}^{N} \int_{\mu_k(0)}^{\mu_k(n)} \tilde{\omega}_\alpha, \quad \tilde{\omega}_\alpha = \frac{\prod_{k\neq\alpha,k=1}^{N}(\lambda-\lambda_k)}{2\sqrt{K(\lambda)P(\lambda)}} d\lambda, \quad \alpha = 1,\dots,N. \tag{8.21}$$

Hence, on the symplectic manifold $(R^{2N}, dE_\alpha \wedge dQ_\alpha)$ the Hamiltonian function $F_0 = \sum_{\alpha=1}^{N} E_\alpha$ produces a linearized flow

$$\begin{cases} \dot{Q}_\alpha = \frac{\partial F_0}{\partial E_\alpha}, \\ \dot{E}_\alpha = 0, \end{cases} \tag{8.22}$$

thus

$$\begin{cases} Q_\alpha(n) = Q_\alpha^0 + t + c_\alpha n, \quad c_\alpha = \sum_{k=1}^{N} \int_{\mu_k(n)}^{\mu_k(n+1)} \tilde{\omega}_\alpha, \\ E_\alpha(n) = E_\alpha(n-1), \end{cases} \tag{8.23}$$

where c_α are dependent on actional variables $\{E_\alpha\}_{\alpha=1}^{N}$, and independent of t; Q_α^0 is an arbitrary fixed constant.

Choose a basic system of closed paths $\alpha_i, \beta_i,\ i = 1,\dots,N$ of Riemann surface $\bar{\Gamma}$: $\mu^2 = P(\lambda)K(\lambda)$ with N handles. $\tilde{\omega}_j$ $(j = 1,\dots,N)$ are exactly N linearly independent holomorphic differentials of the first kind on this Riemann surface $\bar{\Gamma}$. $\tilde{\omega}_j$ are normalized as $\omega_j = \sum_{l=1}^{N} r_{j,l}\tilde{\omega}_l$, i.e. ω_j satisfy

$$\oint_{\alpha_i} \omega_j = \delta_{ij}, \quad \oint_{\beta_i} \omega_j = B_{ij}$$

where $B = (B_{ij})_{N\times N}$ is symmetric and the imaginary part ImB of B is a positive definite matrix.

By Riemann Theorem [25] we know: $\mu_k(n)$ satisfies $\sum_{k=1}^{N} \int_{\mu_k(0)}^{\mu_k(n)} \omega_j = \phi_j,\ \phi_j = \phi_j(n,t) \triangleq \sum_{l=1}^{N} r_{j,l}(Q_l^0 + t + c_l n),\ j = 1,\dots,N$ iff $\mu_k(n)$ are the zero points of the Riemann-Theta function $\tilde{\Theta}(P) = \Theta(A(P)-\phi-K)$ which has exactly N zero points, where $A(P) = (\int_{P_0}^{P}\omega_1,\cdots,\int_{P_0}^{P}\omega_N)^T$, $\phi = \phi(n,t) = (\phi_1(n,t),\cdots,\phi_N(n,t))^T$, $K \in \mathbf{C}^N$ is the Riemann constant vector, P_0 is an arbitrary given point on Riemann surface $\bar{\Gamma}$.

Because of [18]

$$\frac{1}{2\pi i}\oint_\gamma \lambda d\ln\tilde{\Theta}(P) = C_1(\bar{\Gamma}) \tag{8.24}$$

where the constant $C_1(\bar{\Gamma})$ has nothing to do with ϕ; γ is the boundary of simple connected domain obtained through cutting the Riemann surface $\bar{\Gamma}$ along closed paths α_i, β_i. Thus, we have a key equality

$$\sum_{k=1}^{N} \mu_k(n) = C_1(\bar{\Gamma}) - Res_{\lambda=\infty_1}\lambda d\ln\tilde{\Theta}(P) - Res_{\lambda=\infty_2}\lambda d\ln\tilde{\Theta}(P) \tag{8.25}$$

where $\infty_1 := (0, \sqrt{P(z^{-1})K(z^{-1})}|_{z=0})$, $\infty_2 := (0, -\sqrt{P(z^{-1})K(z^{-1})}|_{z=0})$. Through a lengthy careful calculation and combining (8.17), we obtain

$$< q_n(t), q_n(t) >= \sum_{\alpha=1}^{N} \lambda_\alpha - C_1(\bar{\Gamma}) + \frac{d}{dt}\left(\ln \frac{\Theta(\phi(n,t)+K+\eta_1)}{\Theta(\phi(n,t)+K+\eta_2)}\right) \tag{8.26}$$

where the j-th component of η_i $(i=1,2)$ is $\eta_{i,j} = \int_{\infty_i}^{P_0} \omega_j$. By the Riemann surface properties, we can also have $\sum_{l=1}^{N} r_{j,l} c_l = \int_{\infty_2}^{\infty_1} \omega_j = \sum_{l=1}^{N} r_{j,l} \int_{\infty_2}^{\infty_1} \tilde{\omega}_l$ which implies $c_l = \int_{\infty_2}^{\infty_1} \tilde{\omega}_l = \int_{P_0}^{\infty_1} \frac{\prod_{i\neq l, i=1}^{N}(\lambda-\lambda_i)}{\sqrt{P(\lambda)K(\lambda)}} d\lambda$. So, the standard Toda equation (8.4) has the following explicit solution, called **algebro-geometric solution**

$$x_n(t) = \ln \frac{\Theta(Un+Vt+Z)}{\Theta(U(n+1)+Vt+Z)} + Cn + Rt + const. \tag{8.27}$$

where $U = \hat{R}\hat{C}$, $V = \hat{R}\hat{J}$, $Z = \hat{R}Q^0 + K + \eta_1$ with $\hat{C} = (c_1, \ldots, c_N)^T$, $\hat{J} = (1, \ldots, 1)^T$, $Q^0 = (Q_1^0, \ldots, Q_N^0)^T$, $R = \sum_{\alpha=1}^{N} \lambda_\alpha - C_1(\bar{\Gamma})$, while matrix $\hat{R} = (r_{j,l})_{N\times N}$ is determined by the relation $\sum_{l=1}^{N} r_{j,l} \oint_{\alpha_i} \tilde{\omega}_l = \delta_{ij}$, and C is certain constant which can be determined by the algebro-geometric properties on the Riemann surface $\bar{\Gamma}$ [16]. The symmetric matrix $B = (B_{ij})_{N\times N}$ in Θ function is determined by $\sum_{l=1}^{N} r_{j,l} \oint_{\beta_i} \tilde{\omega}_l = B_{ij}$.

Hence, the algebro-geometric solution of Toda lattice (8.3) is

$$\begin{cases} u_n(t) = e^{x_{n+1}-x_n} = e^C \cdot \frac{\Theta^2(U(n+1)+Vt+Z)}{\Theta(U(n+2)+Vt+Z)\Theta(Un+Vt+Z)}, \\ v_n(t) = \dot{x}_n = R + \frac{d}{dt} \ln \frac{\Theta(Un+Vt+Z)}{\Theta(U(n+1)+Vt+Z)}. \end{cases} \tag{8.28}$$

Obviously, the algebro-geometric solution $u_n(t)$ and $v_n(t)$ given by (8.28) are quasi-periodic functions, and they are periodic iff $U = \frac{M}{N}$, where M is a N-dimensional integer column vector. It is easy to see that (8.28) is the finite-band solution of Toda lattice (8.3) if $\lambda_1, \ldots, \lambda_N$ are chosen as the eigenvalues of Toda spectral problem (5.13).

8.2 AKNS equation

In subsection 5.3 we have shown that the constrained AKNS flow shares a common r-matrix with the constrained Dirac flow, therefore they are integrable. Now, we derive the algebro-geometric solution for the second order AKNS equation (8.31).

It is well-known that the AKNS hierarchy is given by

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_j} = JG_j, \quad j = 0, 1, 2, \ldots \tag{8.29}$$

where $\{G_j = J^{-1}KG_{j-1}\}_{j=0}^{\infty}$ is the Lenard sequence, $G_{-1} = (0,0)^T \in KerJ$, the two symmetric operators K, J are

$$K = \begin{pmatrix} 2u\partial^{-1}u & \partial - 2u\partial^{-1}v \\ \partial - 2v\partial^{-1}u & -2v\partial^{-1}v \end{pmatrix}, \quad J = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{8.30}$$

A representative equation ($j = 2$) of (8.29) is

$$u_t = -\frac{1}{2}u_{xx} + u^2 v, \ v_t = \frac{1}{2}v_{xx} - v^2 u, \ t = t_2. \tag{8.31}$$

The independent N-involutive system of the constrained AKNS flow is expressed by Eq. (5.41). Similarly, we consider the following Hamiltonian functions

$$\begin{aligned} F_s^{AKNS} &= \sum_{j=1}^{N} \lambda_j^s E_j^{AKNS} \\ &= 2 < \Lambda^s p, q > - \sum_{j+k=s-1} (< \Lambda^j p, p >< \Lambda^k q, q > - < \Lambda^j p, q >< \Lambda^k q, p >). \end{aligned} \tag{8.32}$$

Let $(p(x,t_s), q(x,t_s))^T$ be the involutive solution of the consistent Hamiltonian canonical equations (H_{AKNS}), (F_s^{AKNS}). Then, we have the following theorem.

Theorem 8.2.1 *$u = - < q(x,t_j), q(x,t_j) >$, $v =< p(x,t_j), p(x,t_j) >$, $j = 0,1,2...$, satisfy the higher-order AKNS equations (8.29). Particularly, Eq. (8.31) is solved with the following solution:*

$$u = - < q(x,t_2), q(x,t_2) >, \ v =< p(x,t_2), p(x,t_2) >, \tag{8.33}$$

where $(p(x,t_2), q(x,t_2))^T$ is the involutive solution of the consistent Hamiltonian systems (H_{AKNS}), (F_2^{AKNS}).

In the following procedure we shall express Eq. (8.33) in an explicit form of algebro-geometric solution. To do so, let us rewrite Eq. (5.35) as follows:

$$L^{AKNS} = \begin{pmatrix} A_{AKNS}(\lambda) & B_{AKNS}(\lambda) \\ C_{AKNS}(\lambda) & -A_{AKNS}(\lambda) \end{pmatrix} \tag{8.34}$$

where

$$A_{AKNS}(\lambda) = 1 + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j q_j, \tag{8.35}$$

$$B_{AKNS}(\lambda) = -\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} q_j^2, \tag{8.36}$$

$$C_{AKNS}(\lambda) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} p_j^2. \tag{8.37}$$

$B_{AKNS}(\lambda),\ C_{AKNS}(\lambda)$ can be changed to the following fractional form:

$$B_{AKNS}(\lambda) \equiv -\frac{< q,q > Q_B(\lambda)}{K(\lambda)}, \quad C_{AKNS}(\lambda) \equiv \frac{< p,p > Q_C(\lambda)}{K(\lambda)} \tag{8.38}$$

where

$$\begin{aligned} < q,q > Q_B(\lambda) &= \sum_{j=1}^{N} q_j^2 \prod_{k=1,k\neq j}^{N} (\lambda - \lambda_k), \\ < p,p > Q_C(\lambda) &= \sum_{j=1}^{N} p_j^2 \prod_{k=1,k\neq j}^{N} (\lambda - \lambda_k), \\ K(\lambda) &= \prod_{j=1}^{N} (\lambda - \lambda_j). \end{aligned}$$

Respectively choosing $N-1$ distinct real zero points $\mu_1^B, \ldots, \mu_{N-1}^B$ and $\mu_1^C, \ldots, \mu_{N-1}^C$ of $Q_B(\lambda)$ and $Q_C(\lambda)$ leads to

$$\sum_{k=1}^{N-1} \mu_k^B = A_1 - \frac{< \Lambda q, q >}{< q,q >}, \quad \sum_{k=1}^{N-1} \mu_k^C = A_1 - \frac{< \Lambda p, p >}{< p,p >}, \tag{8.39}$$

$$(A_1 - \sum_{k=1}^{N-1} \mu_k^B)^2 - \sum_{k=1}^{N-1} (\mu_k^B)^2 = 2A_2 - A_1^2 + 2\frac{< \Lambda^2 q, q >}{< q,q >}, \tag{8.40}$$

$$(A_1 - \sum_{k=1}^{N-1} \mu_k^C)^2 - \sum_{k=1}^{N-1} (\mu_k^C)^2 = 2A_2 - A_1^2 + 2\frac{< \Lambda^2 p, p >}{< p,p >}, \tag{8.41}$$

where $A_1 = \sum_{j=1}^{N} \lambda_j,\ A_2 = \sum_{k,j=1,j<k}^{N} \lambda_j \lambda_k$ are two constants. One hand, $u_x = -2 < q, q_x >= -2 < q, \frac{\partial H_{AKNS}}{\partial p} >= -2 < \Lambda q, q > -2uc_0(t)$, here $c_0(t)$ is an arbitrarily fixed function of t. Thus from Eq. (8.40) we have

$$\frac{\partial}{\partial x} \ln u = 2A_1 - 2\sum_{k=1}^{N-1} \mu_k^B - 2c_0(t). \tag{8.42}$$

On the other hand, $u_{t_2} = -2 < q, q_{t_2} >= -2 < q, \frac{\partial F_2^{AKNS}}{\partial p} >= -2 < \Lambda^2 q, q >$. This is combined with Eq. (8.40) to give the equality

$$\frac{\partial}{\partial t_2} \ln u = (A_1 - \sum_{k=1}^{N-1} \mu_k^B)^2 - \sum_{k=1}^{N-1} (\mu_k^B)^2 - 2A_2 + A_1^2. \tag{8.43}$$

So, we obtain

$$\begin{aligned} u(x,t) &= u(x_0,t_0) \exp(\int_{t_0}^{t} \left[(A_1 - \sum_{k=1}^{N-1} \mu_k^B)^2 - \sum_{k=1}^{N-1} (\mu_k^B)^2 - 2A_2 + A_1^2 \right] dt \\ &\quad + \int_{x_0}^{x} \left[2A_1 - 2\sum_{k=1}^{N-1} \mu_k^B - 2c_0(t) \right] dx),\ t = t_2, \end{aligned} \tag{8.44}$$

where $x_0,\ t_0$ are two fixed initial values. Similarly, $v(x,t)$ has the following representation

$$\begin{aligned} v(x,t) &= v(x_0,t_0)\exp(-\int_{t_0}^{t}\left[(A_1-\sum_{k=1}^{N-1}\mu_k^C)^2-\sum_{k=1}^{N-1}(\mu_k^C)^2-2A_2+A_1^2\right]dt \\ &\quad -\int_{x_0}^{x}\left[2A_1-2\sum_{k=1}^{N-1}\mu_k^C-2c_0(t)\right]dx),\ t=t_2. \end{aligned} \tag{8.45}$$

Since Eqs. (8.44) and (8.45) solves nonlinear equation (8.31), then in order to obtain their explicit form it needs calculating the four key expressions $\sum_{k=1}^{N-1}(\mu_k^J)^k$, $J=B,C;\ k=1,2$. For that purpose, we follow the approach in the case of Toda lattice equation. For the present two set of Darboux coordinates $\mu_j^J,\ J=B,C;$ $j=1,...,N-1$, we have the following key equalities like Eq. (8.25)

$$\sum_{j=1}^{N-1}(\mu_j^J)^k = C_k(\Gamma)-\sum_{s=1}^{2}Res_{\lambda=\infty_s}\lambda^k d\ln\Theta(A(P)-\phi-K_J), \tag{8.46}$$
$$J=B,C;\ k=1,...,N-1,$$

where $C_k(\Gamma)$ is a constant [36, 56] only determined by the compact Riemann surface Γ ($genus=N-1$): $\mu^2=P_{AKNS}(\lambda)K(\lambda)$, $P_{AKNS}(\lambda)=K(\lambda)+\sum_{j=1}^{N}E_j^{AKNS}\prod_{k\neq j,k=1}^{N}$ $(\lambda-\lambda_k)$; $\infty_1=(0,\sqrt{P_{AKNS}(z^{-1})K(z^{-1})}|_{z=0})$, $\infty_2=(0,-\sqrt{P_{AKNS}(z^{-1})K(z^{-1})}|_{z=0})$; $A(P)=\int_{P_0}^{P}\omega$ is an Abel map in which P_0 is an arbitrarily fixed point on Γ, $\omega=(\omega_1,...,\omega_{N-1})^T, \omega_j=\sum_{l=1}^{N-1}r_{j,l}\tilde{\omega}_l=\sum_{l=1}^{N-1}r_{j,l}\frac{\prod_{k\neq l,k=1}^{N}(\lambda-\lambda_k)}{2\sqrt{K(\lambda)P_{AKNS}(\lambda)}}d\lambda$ is a normalized holomorphic differential form, and $r_{j,l}$ is the normalized factor; The j-th component $\phi_j(x,t)$ of $N-1$ dimensional vector ϕ equals to $\sum_{l=1}^{N-1}r_{j,l}(Q_l^0+\frac{1}{2}\lambda_l x+\frac{1}{2}\lambda_l^2 t+C_l(t)+\tilde{C}_l(x))$ with the arbitrary constant Q_l^0 and functions $C_l(t),\ \tilde{C}_l(x)\in C^\infty(R)$; $K_B,\ K_C\in\mathbf{C}^{N-1}$ are the two Riemann constant vectors respectively associated with the Darboux coordinates $\mu_j^B,\ \mu_j^C$; Riemann-Theta function [34] $\Theta(\xi)$ is defined on Riemann surface Γ.

A lengthy calculation of Residue at $\infty_s,\ s=1,2$ for $k=1,2$ yields

$$\sum_{j=1}^{N-1}\mu_j^J = C_1(\Gamma)-\frac{\partial}{\partial x}\ln\frac{\Theta_1^J}{\Theta_2^J}, \tag{8.47}$$

$$\sum_{j=1}^{N-1}(\mu_j^J)^2 = C_2(\Gamma)+\frac{\partial}{\partial t}\ln\frac{\Theta_1^J}{\Theta_2^J}-\frac{\partial^2}{\partial x^2}\ln\Theta_1^J\Theta_2^J, \tag{8.48}$$

where $\Theta_s^J=\Theta(\phi+K_J+\eta_s),\ J=B,C,\ \eta_{s,j}=\int_{\infty_s}^{P_0}\omega_j, s=1,2$ is the j-th component of the $N-1$ dimensional vector η_s.

Substituting the above equalities into (8.44) and (8.45), and sorting them, we obtain the explicit solution of nonlinear equation (8.31):

$$u(x,t) = u(x_0,t_0)e^{a(t-t_0)+2(b-c_0(t))(x-x_0)}\frac{\Theta_1^B}{\Theta_2^B}|_{t=t_0}(\frac{\Theta_2^B}{\Theta_1^B})^2|_{x=x_0}$$

$$\times\frac{\Theta_1^B}{\Theta_2^B}\exp(\int_{t_0}^{t}\left[\frac{\partial^2}{\partial x^2}\ln\Theta_1^B\Theta_2^B+(b+\frac{\partial}{\partial x}\ln\frac{\Theta_1^B}{\Theta_2^B})^2\right]dt), \qquad (8.49)$$

$$v(x,t) = v(x_0,t_0)e^{-a(t-t_0)-2(b-c_0(t))(x-x_0)}\frac{\Theta_2^C}{\Theta_1^C}|_{t=t_0}(\frac{\Theta_1^C}{\Theta_2^C})^2|_{x=x_0}$$

$$\times\frac{\Theta_2^C}{\Theta_1^C}\exp(\int_{t_0}^{t}\left[\frac{\partial^2}{\partial x^2}\ln\Theta_2^C\Theta_1^C+(b+\frac{\partial}{\partial x}\ln\frac{\Theta_2^C}{\Theta_1^C})^2\right]dt), \qquad (8.50)$$

where $a = A_1^2 - C_2(\Gamma) - 2A_2,\ b = A_1 - C_1(\Gamma)$ are two constants, $c_0(t) \in C^\infty(R)$ is an arbitrarily given function of t, and $x_0,\ t_0$ are the initial values. Therefore, we have the following theorem.

Theorem 8.2.2 *The AKNS equation (8.31) has the explicit solution (8.49) and (8.50) given by the form of Riemann-Theta function, which is called the* **algebro-geometric solution**.

An analogous calculational process will lead to the algebro-geometric solution of the higher-order AKNS equation (8.29).

9 Conclusions and problems

In finite dimensional case, Lax matrix is enough to provide many important integrable properties like r-marix, Hamiltonian, integrability, Darboux coordinates, and even later algebro-geometric solution. Therefore we specially stress to use Lax matrix instead of Lax pair in finite dimensional case.

The generalized r-matrix structure is given to emphasize the classification and united sketch of finite dimensional integrable systems. We have already seen that only is there one concrete r-matrix structure, then the corresponding Hamiltonian flows are surely integrable and even in some cases the associated spectral problems are new.

In the paper, we develope our generalized structure to solve some integrable equations with algebro-geometric solution. This is an extension of nonlinearization methods [8]. It is found that this procedure can be also applied into other integrable NLEEs [56, 55, 19]. In this sense, we successfully realize a procedure from finite dimensional flows to infinite dimensional systems when we have some constrained or restricted relation between them. Of course, there are still other methods to solve integrable NLEEs. Recently, Deift, Its and Zhou [15, 17] obtained the Θ-function solutions of some integrable NLEEs like the KdV, MKdV, nonlinear Schrödinger equation by using Riemann-Hilbert asymptotic method. All these methods are still under the development.

It should be pointed out that our procedure is carried in the symplectic space $(R^{2N},\ dp\wedge dq)$ (i.e. corresponding to the Bargmann constraint). How about the

case restricted on a subsymplectic manifold in the space R^{2N} (i.e. corresponding to the C. Neumann constraint)? This is a difficult problem. Although r-matrix works out [57], and there is no answer about the algebro-geometric solution up to now.

From section 5, we know that the c-Toda (or r-Toda) flow and the c-CKdV (or r-Toda) flow share the same r-matrix as well as the common Lax matrix and involutive conserved integrals in the whole space R^{2N} (or on certain symplectic submanifolds in R^{2N}). Therefore a further conjecture is: whether any finite dimensional continuous Hamiltonian flow can be associated with a finite dimensional discrete symplectic map such that they share a common Lax matrix? If it is right, then the discrete integrable systems will be mostly enlarged.

Acknowledgments

The author would like to express his sincere thanks to Prof. Gu Chaohao and Prof. Hu Hesheng for their enthusiastic instructions and helps, and thanks Prof. Cao Cewen, Prof. Liu Zhangju and Prof. Strampp for their helpful discussions and advices. He thanks very much for Dr. Holm for his fruitful discussion. He is also very grateful to the Fachbereich 17 of the University Kassel, Germany, in particular to Prof. Strampp and Prof. Varnhorn for their warm invitation and hospitality, and to the referees for their precious suggestions.

This work has been supported by the Alexander von Humboldt Foundation, Germany; the Chinese National Basic Reseach Project "Nonlinear Science", the Special Grant of Chinese National Excellent Ph. D. Thesis, and the Doctoral Programme Foundation of the Institution of High Education, China.

References

[1] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, The inverse scattering transform – Fourier analysis for nonlinear problem, Stud. Appl. Math. 53(1974), 249-315.

[2] M. Adler and P. van Moerbecke, Completely integrable systems, Euclidean Lie algebras and curves, Adv. Math. 38(1980), 267-317.

[3] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadephia, 1981.

[4] M. Antonowicz and S. R. Wojciechowski, Constrained flows of integrable PDEs and bi-Hamiltonian structure of the Garnier system, Phys. Lett.A 147(1990), 455-462.

[5] M. Antonowicz and S. R. Wojciechowski, Restricted flows of soliton hierarchies: coupled KdV and Harry-Dym case, J. Phys. A 24(1991), 5043-5061.

[6] V. I. Arnol'd, Mathematical Methods of Classical (Springer-Verlag, Berlin, 1978).

[7] O. Babelon and C. M. Viallet, Hamiltonian structure and Lax equations, Phys. Lett. B 237(1990), 411-416.

[8] C. W. Cao, Nonlinearization of Lax system for the AKNS hierarchy, Sci. China A (in Chinese) 32(1989), 701-707; also see English Edition: Nonlinearization of Lax system for the AKNS hierarchy, Sci. Sin. A 33(1990), 528-536.

[9] C. W. Cao, Stationary Harry-Dym's equation and its relation with geodesics on ellipsoid, Acta Math. Sin. New Series 6(1990), 35-41.

[10] C. W. Cao, Parametric representation of the finite band solution of the Heisenberg equation, Phys. Lett. A 184(1994), 333-338.

[11] C. W. Cao, A claasical integrable system and the involutive representation of solution of the KdV equation, Acta Math. Sin. New Series 7(1991), 216-223.

[12] C. W. Cao and X. G. Geng, Classical integrable systems generated through nonlinearization of eigenvalue problems, in: Reports in phys, Nonlinear phys, eds. Chaohao Gu, Yishen Li and Guizhang Tu (Springer, Berlin, 1990) pp.68-78.

[13] C. W. Cao and X. G. Geng, C. Neumann and Bargmann systems associated with the coupled KdV soliton hierarchy, J. Phys. A 23(1990), 4117-4125.

[14] E. Date and S. Tanaka, Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice, Prog. Theor. Phys. 55(1976), 457-465.

[15] P. A. Deift, A. R. Its and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, Ann. of Math. 146(1997), 149-235.

[16] P. A. Deift, T. Kriecherbauer and S. Venakides, Forced lattice vibrations: Part II, Comm. Pure Appl. Math. 128(1995), 1251-1298.

[17] P. A. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems and Asymtotics for the MKdV equation, Ann. of Math. 137(1993), 295-368.

[18] L. A. Dickey, Soliton Equations and Hamiltonian Systems (World Scientific, Singapore, 1991).

[19] D. L. Du, "Integrability of the nonlinearized eigenvalue problems under the Lie-Poisson structure", Ph. D. Thesis, 1998, Zhengzhou University, China.

[20] B. A. Dubrovin, V. B. Mateev and S. P. Novikov, Nonlinear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties, Russ. Math. Surv. 31(1976), 59-146.

[21] L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons (Springer, Berlin, 1987).

[22] X. G. Geng, A new hierarchy of nonlinear evolution equations and corresponding finite dimensional completely integrable systems, Phys. Lett. A 162(1992), 375-380.

[23] X. G. Geng, A hierarchy of nonlinear evolution equations, its Hamiltonian structure and classical integrable system, Physica A 180(1992), 241-251.

[24] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-deVries equation, Phys. Rev. Lett. 19(1967), 1095-1097.

[25] P. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley,New York, 1978).

[26] H. Knörrer, Geodesics on quadrics and a mechanical problem of C. Neumann, J. Rein. Ang. Math., 334(1982), 69-78.

[27] I. M. Krichever, Commutative rings of ordinary linaer differential operators, Funct. Anal. Appl. 12(1978), 175-185; Algebraic curves and nonlinear difference equations, Russ. Math. Surv. 33(1978), 255-256.

[28] P. D. Lax, Periodic solutions of the KdV equation, Comm. Pure Appl. Math., 28(1975), 141-188.

[29] D. Levi, A. Sym and S. R. Wojciechowski, A hierarchy of coupled KdV equations and normalization conditions of the Hilbert-Riemann problem, J. Phys. A 16(1983), 2423-2432.

[30] B. M. Levitan and I. S. Sargsjan, Introduction to Spectral Theory: Self-Adjoint Ordinary Differential Operator (American Mathematical Society, Providence, RI, 1975).

[31] Y. S. Li and D. W. Zhuang, Nonlinear evolution equations associated with the eigenvalue problems with energy-dependent potentials, Acta Math. Sin. 25(1982), 464-474.

[32] S. V. Manakov, Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body, Funct. Anal. Appl. 10(1976), 328-329.

[33] J. Moser, Three integrable Hamitonian systems connected with isospectral deformation, Adv. Math. 16(1975), 197-220.

[34] D. Mumford, Tata Lectures on Theta I, II (Birkhäuser,1984).

[35] Z. J. Qiao, "Nonlinearization of Lax pair for the Levi hierarchy and completely integrable system", Master Thesis, Zhengzhou University, P. R. China, June, 1989.

[36] Z. J. Qiao, "Generalized Lax algebra, r-matrix and algebro-geometric soultion for the integrable system", preprint, 1996, Ph. D. Thesis (10246/950011), Fudan University, China.

[37] Z. J. Qiao, Modified r-matrix and separation of variables for the MKdV hierarchy, Physica A 243(1997), 129-140.

[38] Z. J. Qiao, A completely integrable system and the parametric representations of solutions of the WKI hierarchy, J. Math. Phys. 36(1995), 3535-3560.

[39] Z. J. Qiao, Completely integrable system related to a new hierarchy of isospectral evolution equations, Phys. Lett. A 192(1994), 316-322.

[40] Z. J. Qiao, A finite-dimensional integrable system and the involutive solutions of the higher-order Heisenberg spin chain equations, Phys. Lett. A 186(1994), 97-102.

[41] Z. J. Qiao, A hierarchy of nonlinear evolution equations and finite dimensional involutive systems, J. Math. Phys. 35(1994), 2971-2977.

[42] Z. J. Qiao, Bargmann system and the involutive solutions of the Levi hierarchy, J. Phys. A 26(1993), 4407-4417.

[43] O. Ragnisco, Dynamical r-matrices for integrable maps, Phys. Lett. A 198(1995), 295-305.

[44] O. Ragnisco, C. W. Cao and Y. T. Wu, On the relations of the stationary Toda equation and the symplectic map, J. Phys. A 28(1995), 573-588.

[45] A. G. Reyman and M. A. Semenov-Tian-Shansky, Reduction of Hamiltonian systems, affine Lie algebras, Lax equations I, Invent. Math. 54(1979), 81-100.

[46] A. G. Reyman and M. A. Semenov-Tian-Shansky, Group-Theoretical Methods in the Theory of Finite Dimensional Integrable Systems, Dynamical system VII, Springer-Verlag 1994, V. I. Arnol'd & S. P. Novikov (Eds.), Ency. Math. Sci. Vol. 16, pp. 116 – 225.

[47] M. A. Semenov-Tian-Shansky, What is a classical r-matrix? Funct. Anal. Appl. 17(1983), 259-272.

[48] E. K. Sklyanin, Separarion of variables, Prog. Theor. Phys. Suppl. 118(1995), 35-60.

[49] L. A. Takhtajan, Integration of the continuous Heisenberg spin chain through the inverse scattering method, Phys. Lett. A 64(1977), 235-237.

[50] M. Wadati, The exact solution of the modified Korteweg-de Vries equation, J. Phys. Soc. Japan 32(1972), 1681; The modified Korteweg-de Vries equation, J. Phys. Soc. Japan 34(1973)1289-1296.

[51] M. Wadati, K. Konno, and Y. H. Ichikawa, New integrable nonlinear evolution equations, J. Phys. Soc. Japan 47(1979), 1698-1700.

[52] M. Wadati and K. Sogo, Gauge Transformations in Soliton Theory, J. Phys. Soc. Japan 52(1983), 394-398.

[53] S. R. Wojciechowski, Newton representation for stationary flows of the KdV hierarchy, Phys. Lett. A 170(1992), 91-94.

[54] V. E. Zakharov and A. B. Shabat, Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, Sov. Phys. JETP 34(1972), 62-69.

[55] J. S. Zhang, "The generating functional method for the finite dimensional integrable systems", Ph. D. Thesis, 1998, Zhengzhou University, China.

[56] R. G. Zhou, The finite dimensional integrable systems related to the soliton equations, preprint, 1996, Ph. D. Thesis, Fudan University, China.

[57] R. G. Zhou and Z. J. Qiao, On restricted Toda and CKdV flows of Neumann type, Comm. Theor. Phys. 18(1999), to appear.